Assignment 1
Intro to Modern Analysis

1. For a subset A of a set X, let A^c denote the complement of A given by $A^c = X \setminus A$. Let I be an index set, and let $\{A_i\}_{i\in I}$ be a collection of subsets of X indexed by I. Prove or disprove.

(a) We have
$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c.$$

(b) We have
$$\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c.$$

2. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of nonempty subsets of $\mathbb{R}$ which are "nested" in the sense that
$$A_0 \supset A_1 \supset A_2 \supset \cdots .$$
Is the intersection
$$\bigcap_{n\in\mathbb{N}} A_n$$
nonempty? Prove or disprove.

3. Let A denote the subset of $\mathbb{R}$ given by
$$A = \left\{ \frac{|\cos(n)|}{n+1} : n \in \mathbb{N} \right\}.$$
Find with proof the values $\sup A$ and $\inf A$, whenever either exists.

4. Let A be a nonempty subset of $\mathbb{R}$ bounded from above, and let x_0 be an upper bound of A. Show that x_0 is equal to $\sup A$ if and only if for each $\epsilon > 0$ there is an element $x \in A$ such that $x_0 - x < \epsilon$.

5. Let A be a nonempty set of *integers* that is bounded from above. Show that A has a largest element. (Hint: Let $x_0 = \sup A$. First show that x_0 is an integer by showing that if not, then there are integers $m, n \in A$ satisfying $x_0 - 1 < m < n < x_0$, which is an impossible statement [why?]. Then show that x_0 belongs to A.)

6. Let A, B be nonempty subsets of $\mathbb{R}$ each bounded from above. Prove or disprove.

(a) $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.

(b) $\sup(A \cap B) = \inf\{\sup A, \sup B\}$.

7. For a real number t, let $|t|$ denote the absolute value of t defined by
$$|t| = \begin{cases} t & t \geqslant 0 \\ -t & t \leqslant 0 \end{cases}.$$

(a) For a real number t and a nonnegative number a, show that $|t| \leqslant a$ if and only if $-a \leqslant t \leqslant a$.

(b) Let x and y be points of $\mathbb{R}^n$. Show that

$$\big|\|x\| - \|y\|\big| \leqslant \|x - y\| \,.$$