Assignment 4

Intro to Modern Analysis

1. Let X, Y be metric spaces.

(a) Show that the identity map $f : X \to X$ defined by $f(x) = x$ is continuous.

(b) For a point q of Y, show that the constant map $g : X \to Y$ defined by $g(x) = q$ is continuous.

2. Let $f : \mathbb{R}^k \to \mathbb{R}$ denote the function determined by the norm $f(x) = \|x\|$. Show that f is continuous.

3. Let X, Y be metric spaces. Suppose X is equipped with the discrete metric

$$d(p,q) = \begin{cases} 1 & p \neq q \\ 0 & p = q \end{cases}.$$

Show that any function $f : X \to Y$ is continuous.

4. Let X, Y be metric spaces. Suppose X is connected and Y satisfies the property that every singleton set $\{y\}$ is open in Y. Show that a function $f : X \to Y$ is continuous if and only if f is constant. Deduce that any continuous function $\mathbb{R} \to \mathbb{N}$ is constant.

5. Let $f : X \to Y$ be continuous.

(a) For any subset $E \subset X$, show that

$$f(\overline{E}) \subset \overline{f(E)}.$$

Also, find an example where the inclusion is strict.

(b) If E is dense in X and f is surjective, show that $f(E)$ is dense in Y.

(c) Let $g : X \to Y$ be another continuous function, and let $X_0 \subset X$ be a dense subset of X. Show that if $f(x) = g(x)$ for each $x \in X_0$, then $f(x) = g(x)$ for each $x \in X$.

6. Let I denote the unit interval $I = [0,1]$ of $\mathbb{R}$. Show that any continuous map $f : I \to I$ has a fixed point, that is, a point $x_0 \in I$ such that $f(x_0) = x_0$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = (x+1)^2$. Let $\epsilon > 0$ be given.

(a) Find a $\delta > 0$ such that if x satisfies $|x - 3| < \delta$, then $|f(x) - f(3)| < \epsilon$.

(b) Find a function $\delta : \mathbb{R} \to \mathbb{R}_{>0}$ such that if $x, p \in \mathbb{R}$ satisfy $|x - p| < \delta(p)$, then $|f(x) - f(p)| < \epsilon$. (Note that part (a) determines a possible value for $\delta(3)$.)

(c) Is it possible to choose $\delta(p)$ from (b) to be independent of p, that is, to be a constant function? Why or why not?

(d) What if the domains of f and δ are restricted to $[-2,0]$? Then is it possible to make $\delta(p)$ constant? Why or why not?

8. Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be the square root function $f(x) = \sqrt{x}$. Show that f is uniformly continuous (even though the domain of f is not compact).

9. Let $f : \mathbb{R} \to \mathbb{R}$ denote the function defined by

$$f(x) = \begin{cases} \frac{1}{n} & x = m/n \text{ for } m, n \text{ relatively prime integers with } n > 0 \\ 0 & x \text{ is irrational} \end{cases}.$$

(And when $x = 0$, take $n = 1$.) Prove that f is continuous at every irrational number and discontinuous at every rational number.

10. Let α be a positive irrational number. Let E denote the subset of $\mathbb{R}$ given by

$$E = \{m + n\alpha : m, n \in \mathbb{Z}\}.$$

The goal of this problem is to show that E is dense in $\mathbb{R}$.

(a) Show that if $e \in E$, then $-e \in E$.

(b) Let $\lfloor \alpha \rfloor$ denote the largest nonnegative integer smaller than α. In other words,

$$\lfloor \alpha \rfloor = \sup(\mathbb{Z} \cap (-\infty, \alpha]).$$

Note that $0 \leqslant \alpha - \lfloor \alpha \rfloor < 1$. For each positive integer k, let

$$\beta_k = k\alpha - \lfloor k\alpha \rfloor .$$

Show that if $j \neq k$, then $\beta_j \neq \beta_k$.

(c) Let N be an integer satisfying $N \geqslant 2$. For each integer ℓ, let

$$A_\ell = \left[\frac{\ell}{N}, \frac{\ell+1}{N}\right).$$

Show that there is an integer ℓ satisfying $0 \leqslant \ell \leqslant N-1$ and integers j, k satisfying $1 \leqslant j < k \leqslant N+1$ such that

$$\beta_j, \beta_k \in A_\ell.$$

(d) Use (b) to show that there is an element $e \in E$ such that $0 < e < \frac{1}{N}$.

(e) For each integer $\ell \geqslant 0$, show that there is an element of E in A_ℓ. Deduce from (a) that there is also an element in $A_{-\ell}$.

(f) For each point $x \in \mathbb{R}$ and each $\epsilon > 0$, show that there is a point in the intersection $E \cap B_\epsilon(x)$.

(g) Deduce that E is dense in $\mathbb{R}$.