Assignment 5
Intro to Modern Analysis

1. Let n be a given fixed positive integer. Find a function such that $f^{(n-1)}$ is continuous, but $f^{(n)}$ is not.

2. Let f be continuous on $[a,b]$ and differentiable on (a,b). Suppose that $f'(x) \neq 0$ for each x in (a,b). Show that f is injective on $[a,b]$.

3. Suppose the derivative f' is continuous on (a,b) and $f'(x) \neq 0$ for each x in (a,b).

(a) Show that f admits an inverse g defined on the image of f.

(b) Prove that g is differentiable and the derivative satisfies

$$g'(f(x)) = \frac{1}{f'(x)}$$

for each $x \in (a,b)$.

4. Let f be continuous on $[a,b]$ and differentiable on (a,b). Set $f(a) = y$ and suppose that $|f'(x)| \leqslant M$ for each $x \in (a,b)$. How large can $f(b)$ be? How small can $f(b)$ be? Prove that the values you find are actually achieved by demonstrating two functions which achieve them.

5. If n is a positive integer and $0 \leqslant y \leqslant x$, show that

$$ny^{n-1}(x-y) \leqslant x^n - y^n \leqslant nx^{n-1}(x-y).$$

6. Suppose f is continuous on $[a,b]$ and $f(x) \geqslant 0$ for each $x \in [a,b]$. Show that if

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0$ for each $x \in [a,b]$.

7. Define f on $[a,b]$ by

$$f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}.$$

Show that f is not Riemann integrable on $[a,b]$.

8. Let f be defined on $(0,1]$. Suppose that f is Riemann integrable on $(c,1]$ for each $c \in (0,1)$.

(a) If f is Riemann integrable on $[0,1]$, show that

$$\int_0^1 f(x)\,dx = \lim_{c \to 0} \int_c^1 f(x)\,dx.$$

(b) Construct a function f for which the limit in (a) exists, even though the same limit fails to exist for $|f|$ in place of f.

9. Let p and q be positive real numbers satisfying

$$\frac{1}{p} + \frac{1}{q} = 1.$$

(a) Show that for any nonnegative numbers $u, v \geqslant 0$, we have

$$uv \leqslant \frac{u^p}{p} + \frac{v^q}{q}.$$

(b) Show that if f and g are both nonnegative (i.e. $f \geqslant 0$ and $g \geqslant 0$) and satisfy

$$\int_a^b f(x)^p \, dx = \int_a^b g(x)^q \, dx = 1,$$

then

$$\int_a^b f(x)g(x) \, dx \leqslant 1.$$

(c) Show that for any two functions (not necessarily nonnegative), we have

$$\left| \int_a^b f(x)g(x) \, dx \right| \leqslant \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} \left(\int_a^b |g(x)|^q \, dx \right)^{1/q}$$

provided both sides make sense. This is called Hölder's inequality.

10. Prove directly from the definitions that if f is Riemann integrable on $[a, b]$ and c is any positive constant, then cf is also integrable on $[a, b]$ and moreover

$$\int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx.$$