Assignment 6
Intro to Modern Analysis

1. Suppose f_n and g_n are sequences of functions converging uniformly on E.

(a) Show that $f_n + g_n$ converges uniformly on E.

(b) If f_n and g_n are bounded, show that $f_n g_n$ converges uniformly on E.

(c) Find an example where $f_n g_n$ does not converge uniformly on E.

2. Suppose f_n is an equicontinuous sequence of functions on a compact set K and f_n converges pointwise on K. Show that f_n converges uniformly on K.

3. Let A be a bounded subset of the space $C([a,b])$ of continuous and bounded real-valued functions on $[a,b]$ with the sup norm. Show that the set of all functions $F : [a,b] \to \mathbb{R}$ of the form

$$F(x) = \int_a^x f(t)\,dt$$

for $f \in A$ is uniformly bounded and equicontinuous.

4. Let X and Y be two metric spaces, and assume that X and Y are compact. Let $M_{X,Y}$ denote the collection of all mappings $f : X \to Y$.

(a) Prove that the function $d : M_{X,Y} \times M_{X,Y} \to \mathbb{R}$ defined by

$$d(f,g) = \sup_{x \in X} d_Y(f(x), g(x))$$

is a metric on $M_{X,Y}$.

(b) Let $C_{X,Y}$ denote the subset of *continuous* mappings. Prove that $C_{X,Y}$ is closed in $M_{X,Y}$. (Hint: At some point you need to show that the limit of a uniformly convergent sequence of mappings is a continuous mapping.)

5. Let A be a subset of $C([a,b])$. Suppose that

(i) A is uniformly bounded

(ii) there is a constant $M > 0$ such that $|f'(x)| \leqslant M$ for each $f \in A$ and each $x \in [a,b]$.

Show if f_n is any sequence in A, then there is a subsequence f_{n_k} that converges uniformly on $[a,b]$.

6. Let $X = C([0,1])$ be the space of continuous and bounded real-valued functions on $[0,1]$ together with the supremum norm. Let A be the unit ball of X given by

$$A = \{f \in X : \sup_{t \in [0,1]} |f(t)| \leqslant 1\}.$$

(a) Show that A is closed and bounded in X.

(b) Show that A is not compact. (Hint: Find a sequence $f_n \in A$ that admits no convergent subsequence. Example 7.21 of Rudin might be helpful.)

7. Recall the space m of bounded sequences of real numbers together with the metric

$$d(x, y) = \sup_{k=1,2,\ldots} |x_k - y_k|.$$

(a) Give a simple proof to show that m is complete by showing that $m = C(X)$ for some suitable space X. (Recall that $C(X)$ denotes the space of continuous bounded real-valued functions on X together with the supremum norm.)

(b) Let A denote the unit ball in m given by

$$A = \{x \in m : \sup_{k=1,2,\ldots} |x_k| \leqslant 1\}.$$

Show that A is not compact. (Hint: Consider the idea of Problem 7(b).)

8. More generally, let X be any infinite-dimensional vector space equipped with an inner product $\langle -, - \rangle$ in such a way that the induced metric is complete. In particular, there is a norm on X defined by

$$\|x\| = \sqrt{\langle x, x \rangle}$$

and the metric is given by

$$d(x, y) = \|x - y\|.$$

Let A denote the unit ball

$$A = \{x \in X : \|x\| \leqslant 1\}.$$

We know that A is closed and bounded essentially from the definitions. Show that A is not compact. (Hint: Construct a sequence $x_n \in A$ as follows. Pick $x_1 \in A$ such that $\|x_1\| = 1$. Once $x_1, \ldots, x_n$ are chosen, use Graham-Schmidt to find an $x_{n+1} \in A$ such that $\|x_{n+1}\| = 1$ and x_{n+1} is orthogonal to each $x_1, \ldots, x_n$. Argue that for distinct m, n we have $\|x_n - x_m\| \geqslant 1/2$. The polarization identity

$$\langle x, y \rangle = \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x - y\|^2)$$

might be useful.)