Mathematics G4402. Modern Geometry I, Fall 2015
Lecture Notes

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Contents

1. Wednesday, September 9, 2015

Abstract manifolds

Definition 1.1 (topological manifolds)**.** A *topological n-manifold* (or a *topological manifold of dimension n*) is a topological space M which is locally homeomorphic to $\mathbb{R}^n$, that is, for each $p \in M$, there is an open neighborhood U of p in M and a homeomorphism ϕ from U to an open set Ω in $\mathbb{R}^n$. We call such a pair (U, ϕ) a *chart* (or *coordinate system*) for M around p, and U is called a *coordinate neighborhood* at p.

Remark 1.2 (cf. [Bo, page 6], [dC, page 29-30])**.** Some textbooks require that the topology of M satisfy the following *additional* two properties.

(i) The topology of M is Hausdorff. Recall that, a topologial space M is Hausdorff if for any two distinct points p and q in M, there exist open sets U and V in M such that $p \in U$, $q \in V$, and $U \cap V$ is empty.

(ii) The topology of M has a countable basis of open sets.
Recall that a collection $\mathcal{B}$ of open subsets in a topological space M is a basis of open sets of M if every open subset of M can be written as a union of elements of $\mathcal{B}$.

Example 1.3 (a non-Hausdorff manifold)**.** Let $M = \mathbb{R} \sqcup \{p\}$ be the disjoint union of the real line $\mathbb{R}$ and a point p. Define a topology on M by the topology generated by open subsets of $\mathbb{R}$ and sets of the form $(U \setminus \{0\}) \cup \{p\}$, where U is an open neighborhood of 0 in $\mathbb{R}$. Note that any neighborhoods of p and 0 intersect, so M is a non-Hausdorff topological space.

For any $q \in \mathbb{R} = M \setminus \{p\}$, $\mathbb{R} \subset M$ is an open neighborhood of q in M, and the identity map $\mathbb{R} \to \mathbb{R}$ is a homemorphism from $\mathbb{R}$ to $\mathbb{R}$. The set $U = (\mathbb{R} \setminus \{0\}) \cup \{p\}$ is an open neighbhorhood of p in M, and[]../Lecture01.pdf

the map $\phi : U \to \mathbb{R}$ given by $\phi(x) = x$ for $x \in \mathbb{R} \setminus \{0\}$ and $\phi(p) = 0$ is a homeomorphism. Therefore, M is a topological 1-manifold.

Example 1.4. An example of a topological manifold which does not have a countable basis is the *long line*. A proper discussion of this manifold would be quite lengthy and would require a digression on set theory, so we choose not to discuss this example further here.

Definition 1.5 (atlas)**.** An *atlas* of a topological n-manifold M is a collection $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ of charts such that the collection $\{U_\alpha : \alpha \in I\}$ is an open cover of M. The maps $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are called *transition functions*.

Remark 1.6.
- I is some index set, which can be finite, countably infinite, or uncountably infinite.
- If follows from the definitions that the transition functions are homeomorphisms.
- If M has a countable atlas, then M has a countable basis of open sets.

Definition 1.7 (C^k atlas)**.** Let k be a positive integer or ∞. A C^k*-atlas* for an n-manifold M is an atlas $\Phi = \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ such that all transition functions are C^k diffeomorphisms of open subsets of $\mathbb{R}^n$.

Definition 1.8. We say that two C^k-atlases $\Phi = \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ and $\Psi = \{(\psi_\beta, V_\beta) : \beta \in J\}$ for a topological manifold M are *equivalent* if their union is a C^k-atlas. A C^k *differentiable structure* on a topological manifold M is a choice of an equivalence class of C^k-atlases. A C^k *manifold* is a topological manifold equipped with a C^k-structure.

A C^∞ differentiable structure is also called a *smooth structure*, and a C^∞ manifold is also called a *smooth manifold*.

Example 1.9. Let k be a positive integer. We endow $M = \mathbb{R}$ with two non-equivalent C^katlases. For the first atlas, take $\Phi = \{(\mathbb{R}, \phi)\}$ where $\phi(x) = x$. For the second atlas, take $\Psi = \{(\mathbb{R}, \psi)\}$ where $\psi(x) = x^3$. Let k be any postive integer, or ∞. Both Φ and Ψ are C^k-atlases since all of their transition functions (which consist of simply the identity map) are C^k-differentiable. However, their union $\Phi \cup \Psi$ is not a C^k-atlas, since the transition function $\phi \circ \psi^{-1}(x) = x^{1/3}$ is not C^k-differentiable.

Example 1.10 (The real projective space $P_n(\mathbb{R})$)**.**
1. As a *set*, $P_n(\mathbb{R})$ is the set of one-dimensional $\mathbb{R}$-linear subspace of $\mathbb{R}^{n+1}$.

2. *Topology.*
Define a surjective map $\pi : \mathbb{R}^{n+1} \backslash \{0\} \to P_n(\mathbb{R})$ by sending a nonzero vector in $\mathbb{R}^{n+1}$ to the one-dimensional $\mathbb{R}$-linear subspace of $\mathbb{R}^{n+1}$ spanned by that vector. For any nonzero vector $x = (x_1, \dots, x_{n+1})$ in $\mathbb{R}^{n+1}$ we let $[x_1, \dots, x_{n+1}]$ denote its image in $P_n(\mathbb{R})$. Note that $[x_1, \dots, x_{n+1}] = [y_1, \dots, y_{n+1}]$ if and only if $(y_1, \dots, y_{n+1}) = \lambda(x_1, \dots, x_{n+1})$ for some nonzero $\lambda \in \mathbb{R}$. Equip the set $P_n(\mathbb{R})$ with the quotient topology determined by the map π. This means that a subset U of $P_n(\mathbb{R})$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$.

Let $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\} \subset \mathbb{R}^{n+1}$ be the unit sphere with center at the origin, equipped with the subset topology. Then $\pi|_{S^n} : S^n \to P_n(\mathbb{R})$ is a covering map of degree 2. The quotient topology determined by $\pi|_{S^n} : S^n \to P_n(\mathbb{R})$ agrees with the quotient topology determined by $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to P_n(\mathbb{R})$. It is easy to see that the quotient topology determined by $\pi|_{S^n}$ is compact and Hausdorff.

3. *Atlas.*
For each positive integer i satisfying $1 \le i \le n+1$, let U_i denote the subset of $P_n(\mathbb{R})$ given by

$$U_i = \{[x_1, \dots, x_{n+1}] \in P_n(\mathbb{R}) : x_i \neq 0\}.$$

Note that U_i is an open subset of $P_n(\mathbb{R})$ since the set $\pi^{-1}(U_i)$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$. Also note that the collection $\{U_i : 1 \le i \le n+1\}$ forms an open cover of $P_n(\mathbb{R})$. Let $\widetilde{\phi}_i : \pi^{-1}(U_i) \to \mathbb{R}^n$ denote the map given by

$$\widetilde{\phi}_i(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i} \right).$$

Note that $\widetilde{\phi}_i$ satisfies $\widetilde{\phi}_i(\lambda x) = \widetilde{\phi}_i(x)$ for each $x \in \pi^{-1}(U_i)$ and each scalar $\lambda \in \mathbb{R}$. It follows that $\widetilde{\phi}_i$ induces a well-defined map $\phi_i : U_i \to \mathbb{R}^n$ described by $\widetilde{\phi}_i = \phi_i \circ \pi$. Since $\widetilde{\phi}_i$ is continuous, we see that ϕ_i is continuous as well. The map $\phi_i^{-1} : \mathbb{R}^n \to U_i$ given by

$$\phi_i^{-1}(x_1, \dots, x_n) = [x_1, \dots, x_{i-1}, 1, x_i, \dots, x_n]$$

is the inverse of $\phi_i : U_i \to \mathbb{R}^n$. The map ϕ_i^{-1} is also continuous since it can be written as the composition $\phi_i^{-1} = \pi \circ s_i$ where $s_i : \mathbb{R}^n \to \mathbb{R}^{n+1} \backslash \{0\}$ is the continuous map given by

$$s_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, 1, x_i, \dots, x_n).$$

It follows that $\phi_i : U_i \to \mathbb{R}^n$ is a homeomorphism.
Therefore the topogical space $P_n(\mathbb{R})$ is a topological n-manifold, and $\Phi = \{(U_i, \phi_i) : i = 1, \dots, n+1\}$ is an atlas on $P_n(\mathbb{R})$.

4. *Transition functions.*

$$\phi_2 \circ \phi_1^{-1}(y_1, \dots, y_n) = \phi_1([1, y_1, \dots, y_n]) = (\frac{1}{y_1}, \frac{y_2}{y_1}, \dots, \frac{y_n}{y_1})$$

$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) = (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n-1} \to \phi_2(U_1 \cap U_2) = (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n-1}$ is a C^∞ diffeomorphism.
The general case $\phi_j \circ \phi_i^{-1}$ $(i \neq j)$ is similar.
Therefore $\Phi = \{(U_i, \phi_i) : i = 1, \ldots, n+1\}$ is a C^∞ atlas on $P_n(\mathbb{R}^n)$, an defines a C^∞ differentible structure on $P_n(\mathbb{R}^n)$. $(P^n(\mathbb{R}), \Phi)$ is a C^∞ n-manifold.

Remark 1.11. Note that the transition functions $\phi_j \circ \phi_i^{-1}$ are real analytic (C^ω), so Φ is indeed a real analytic atlas, and $(P^n(\mathbb{R}), \Phi)$ is a real analytic manifold of dimension n.

Remark 1.12. Replacing $\mathbb{R}$ by $\mathbb{C}$ in Example 1.10, we obtain the definition of the n-dimensional complex projective space $P_n(\mathbb{C})$, equipped with the quotient topology determined by $\pi : \mathbb{C}^{n+1} - \{0\} \to P_n(\mathbb{C})$. $P_n(\mathbb{C})$ is locally homeomorphic to $\mathbb{C}^n = \mathbb{R}^{2n}$, so it is a topological $2n$-manifold. $\Phi = \{(U_i, \phi_i) : i = 1, \ldots, n+1\}$, where $\phi_i : U_i \to \mathbb{C}^n = \mathbb{R}^{2n}$, is a C^∞ atlas on $P_n(\mathbb{C})$, and $(P_n(\mathbb{C}), \Phi)$ is a C^∞ $2n$-manifold.

The transition functions $\phi_j \circ \phi_i^{-1}$ are indeed complex analytic, so Φ defines a complex struture on $P_n(\mathbb{C})$, and $(P_n(\mathbb{C}), \Phi)$ is a complex manifold of dimensiona n. (cf. Phong's class "Complex Analysis and Riemann Surfaces")

2. Monday, September 14, 2015

C^k-differentiable maps

Definition 2.1. Let M and N be C^l-manifolds of dimension m and n respectively. A continuous map $f : M \to N$ is called *C^k-differentiable* for some $k \leq l$ if for any $p \in M$, there is a coordinate chart (U, ϕ) around p in some atlas representing the C^l-structure on M and a coordinate chart (V, ψ) around $f(p)$ in some atlas representing the C^l-structure on N such that

- $f(U) \subset V$
- the composition $g = \psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is C^k-differentiable.

Remark 2.2. There are two subtleties to this definition.

- The definition seems to depend on choices of coordinate charts in fixed atlases for M and N respectively. Indeed, one might worry that while the $g = \psi \circ f \circ \phi^{-1}$ is C^k-differentiable, there is another such composition $\widetilde{g} = \widetilde{\psi} \circ f \circ \widetilde{\phi}^{-1}$ that is not. However, because the transition maps in a C^l atlas are C^l-differentiable and $k \leq l$, the chain rule forbids this from happening. It follows that the definition does not depend on the choices of coordinate charts in fixed atlas for M and N.
- One might worry, nevertheless, that the definition depends on the choice of atlases representing the given C^l-structures. But again, because of the equivalence condition we placed on C^l-atlases, we see that the chain rule guarantees that the definition does not depend on the choice of atlases representing the given C^l-structures.

These subtleties will appear in forthcoming definitions as well, but we will neglect to remark on them and leave the details to the interested reader.

Definition 2.3. A C^∞-differentiable map $f : M \to N$ is also called a *smooth map*.

Example 2.4. As an example, let us view $\mathbb{R}^{n+1} \setminus \{0\}$ as a smooth manifold where the C^∞-structure is the one determined by the atlas consisting only of the identity map, and let us equip $P_n(\mathbb{R})$ with the C^∞-structure described in Example 1.10. Then the natural map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to P_n(\mathbb{R})$ is a smooth map. This can be seen because the compositions

$$g_i := \phi_i \circ \pi \circ \mathrm{id}^{-1} : \pi^{-1}(U_i) \to \mathbb{R}^n, \quad (x_1, \ldots, x_{n+1}) \mapsto (\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i})$$

are smooth at each point of their domains.

Remark 2.5. If M is a C^l manifold and U is an open subset of M, then the C^l-differentiable structure on M restricts to a C^l-differentiable structure on U.

Definition 2.6. Let M, N be smooth manifolds. We say that $f : M \to N$ is a *diffeomorphism* if

- f is a homeomorphism, and
- f and f^{-1} are smooth.

We say that f is a *local diffeomorphism* at $p \in M$ if there is an open neighborhood U of p in M and an open neighborhood V of $f(p)$ in N such that $f(U) = V$ and $f|_U : U \to V$ is a diffeomorphism.

Example 2.7. Let $\phi : \mathbb{R} \to \mathbb{R}$ be the map $\phi(x) = x$ and let $\psi : \mathbb{R} \to \mathbb{R}$ be the map $\psi(x) = x^3$. We have seen that $\Phi = \{(\mathbb{R}, \phi)\}$ and $\Psi = \{(\mathbb{R}, \psi)\}$ are two C^∞ atlases on $\mathbb{R}$ which are not equivalent. Let $f : (\mathbb{R}, \Phi) \to (\mathbb{R}, \Psi)$ denote the map $f(x) = x^{1/3}$. Then f is a diffeomorphism since $\psi \circ f \circ \phi^{-1} : \phi(\mathbb{R}) = \mathbb{R} \to \Psi(\mathbb{R}) = \mathbb{R}$ is the identity map.

Definition 2.8. Given an open subset U of $\mathbb{R}^m$ and a smooth map $f : U \to \mathbb{R}^n$, we say that f is a *submersion* (resp. *immersion*) at $x \in U$ if the differential $df_x : \mathbb{R}^m \to \mathbb{R}^n$ is a surjective (resp. injective) linear map.

Example 2.9 (Canonical submersion)**.** Let m and n be positive integers satisfying $m \geq n$. Consider the map $\pi : \mathbb{R}^m \to \mathbb{R}^n$ given by

$$\pi(x_1, \ldots, x_m) = (x_1, \ldots, x_n).$$

Since π is a linear map, we see that $d\pi_x = \pi$ for each $x \in \mathbb{R}^m$. It follows that π is a submersion at any $x \in \mathbb{R}^m$; π is called the *canonical submersion.*

Example 2.10 (Canonical immersion)**.** Let m and n be positive integers satisfying $m \leq n$. Consider the map $i : \mathbb{R}^m \to \mathbb{R}^n$ given by

$$i(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0).$$

Since i is a linear map, we have $di_x = i$ for each $x \in \mathbb{R}^m$. It follows that i is an immersion at any $x \in \mathbb{R}^m$; i is called the *canonical immersion.*

Definition 2.11. Let $f : M \to N$ be a smooth map between smooth manifolds and let p be a point of M. We say that f is a *submersion* (resp. *immersion*) at p if there is a chart (U, ϕ) for M around p and a chart (V, ψ) for N around $f(p)$ such that

- $f(U) \subset V$, and
- the composition $g = \psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is a submersion (resp. immersion) at $\phi(p)$.

Proposition 2.12. *Let $f : M \to N$ be a smooth map between smooth manifolds of dimension m and n respectively.*

(1) *(Canonical form for submersions and immersions) If f is a submersion (resp. immersion) at $p \in M$, so that $m \geq n$ (resp. $m \leq n$), there is a chart (U, ϕ) for M around p and a chart (V, ψ) for N around $f(p)$ such that*
- $\phi(p) = 0 \in \mathbb{R}^m$,
- $\psi(f(p)) = 0 \in \mathbb{R}^n$, *and*
- *the composition $\psi \circ f \circ \phi^{-1}$ is the restriction of the canonical submersion (resp. immersion) to $\phi(U) \subset \mathbb{R}^m$.*

(2) *If f is a submersion and an immersion at $p \in M$, then f is a local diffeomorphism at p.*

Proof. Roundtable on September 18. Reference: [Bo, II.7, III.4]. □

Definition 2.13. Let $f : M \to N$ be a smooth map between smooth manifolds. We say that f is a *submersion* (resp. *immersion*) if f is a submersion (resp. immersion) at each point $p \in M$.

Definition 2.14. Let $f : M \to N$ be a smooth map between smooth manifolds. We say that f is an *embedding* if
- f is an immersion
- $f : M \to f(M)$ is a homeomorphism onto $f(M)$, where $f(M)$ is equipped with the subspace topology.

In this case, we say that $f(M)$ is a *submanifold* of N.

From Proposition 2.12 (1), We also have the following alternative definition of a submanifold.

Definition 2.15. Let N be a smooth n-dimensional manifold, and let M be a subset of N. We say that M is a *submanifold* of N of dimension m (which is not greater than n) if for each p in M, there is a chart (U, ϕ) for N around p such that $\phi(p) = 0$ and $\phi(U \cap M) = \phi(U) \cap (\mathbb{R}^m \times \{0\})$.

Example 2.16. These examples are to illuminate the definition of an embedding. Given a smooth map $f : \mathbb{R} \to \mathbb{R}^2$, $df_t : \mathbb{R} \to \mathbb{R}^2$ is given by $df_t(u) = f'(t)u$. So f is an immersion at $t \in \mathbb{R}$ iff $f'(t)$ is nonzero.

(1) Let $f : \mathbb{R} \to \mathbb{R}^2$ denote the parabola given by $f(t) = (t, t^2)$. Then $f'(t) = (1, 2t)$ is nonzero for any $t \in \mathbb{R}$, and hence f is an immersion. We see also that f is a homeomorphism from $\mathbb{R}$ onto the image $f(\mathbb{R})$, so f defines an embedding.
(2) Let $f : \mathbb{R} \to \mathbb{R}^2$ denote the covering of the unit circle given by $f(t) = (\cos(t), \sin(t))$. Then $f'(t) = (-\sin t, \cos t)$ is nonzero for any $t \in \mathbb{R}$, so f is an immersion, but f is not an embedding because it is not injective.
(3) Let $f : \mathbb{R} \to \mathbb{R}^2$ be the nodal cubic defined by $f(t) = (t^3 - 4t, t^2 - 4)$. Then $f'(t) = (3t^2 - 4, 2t)$ is always nonzero, so f is an immersion. However, f is not an embedding since it is not injective: $f(2) = f(-2) = (0, 0)$.
(4) Let $f : \mathbb{R} \to \mathbb{R}^2$ be the cuspidal cubic defined by $f(t) = (t^3, t^2)$. Then we see that f is injective and a homeomorphism onto its image, but f is not an immersion at $t = 0$, because the derivative vanishes there.

Definition 2.17. Let $f : M \to N$ be a smooth map between smooth manifolds and assume that the dimension of M is greater than or equal to the dimension of N. A point $p \in M$ is a *critical point* of f if f is not a submersion at p. In this case, $f(p)$ is called a *critical value* of f, that is, a point $q \in N$ is a critical value if there is a point $p \in f^{-1}(q)$ such that p is a critical point. We say that $q \in N$ is a *regular value* if q is not a critical value.

Theorem 2.18. *Let $f : M \to N$ be a smooth map between smooth manifolds of dimensions m and n respectively, with $m \geq n$. If $q \in N$ is a regular value of f then the preimage $f^{-1}(q)$ is a closed submanifold of M of dimension $m - n$. ($f^{-1}(q)$ can be empty.)*

Proof. Roundtable on September 18. Reference: [Bo, III.5]. Idea: use canonical form of submersion. □

Example 2.19. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be the smooth map given by

$$f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2.$$

Then $df_x : \mathbb{R}^{n+1} \to \mathbb{R}$ is given by $df_x = [2x_1 \cdots 2x_{n+1}]$, which is surjective iff $x \neq 0$. So the only critical point of f is $0 \in \mathbb{R}^{n+1}$ and the only critical value of f is $0 \in \mathbb{R}$. It follows that every nonzero real number is a regular value of f. If $a > 0$, then we see that $f^{-1}(a)$ is a n-dimensional smooth submanifold of $\mathbb{R}^{n+1}$. Note that $f^{-1}(a)$ is the n-dimensional sphere of radius $\sqrt{a}$. We have $f^{-1}(0) = \{0\}$, and $f^{-1}(a)$ is empty when $a < 0$.

Example 2.20. Let p denote the composition $S^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\} \to P_n(\mathbb{R})$. Then p is a covering map of degree 2. Moreover, p is a local diffeomorphism.

3. Wednesday, September 16, 2015

Example 3.1. Let $O(n)$ denote the set of all $n \times n$ orthognal matrices:

$$O(n) = \{A \in M_n(\mathbb{R}) : AA^T = I_n\}$$

where $M_n(\mathbb{R})$ is the set of real $n \times n$ matrics, A^T is the transpose of A, and I_n denotes the $n \times n$ identity matrix. We may identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$ as an n^2-dimensional real vector space. We claim that $O(n)$ is a submanifold of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ of dimension $\frac{n(n-1)}{2}$. To prove this, we will use the preimage theorem.

Let $S_n(\mathbb{R})$ denote the set of all real symmetric $n \times n$ matrices:

$$S_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A = A^T\}.$$

Then $S_n(\mathbb{R})$ is an $\frac{n(n+1)}{2}$-dimensional subspace of $M_n(\mathbb{R})$. Define a map

$$f : M_n(\mathbb{R}) \cong \mathbb{R}^{n^2} \longrightarrow S_n(\mathbb{R}) \cong \mathbb{R}^{\frac{n(n+1)}{2}}, \quad A \mapsto AA^T.$$

Then f is a smooth map, since it is a polynomial map in the entries of A: if $A = (a_{ij})$ then $(AA^T)_{kl} = \sum_{m=1}^n a_{km}a_{lm}$.

By the preimage theorem, it remains to show that I_n is a regular value of f. For $A \in M_n(\mathbb{R})$, the differential $df_A : M_n(\mathbb{R}) \to S_n(\mathbb{R})$ at A is given by

$$df_A(B) = \lim_{h \to 0} \frac{f(A + hB) - f(A)}{h} = \lim_{h \to 0} \frac{(A + hB)(A^T + hB^T) - AA^T}{h} = AB^T + BA^T.$$

If $A \in f^{-1}(I_n) = O(n)$ and $C \in S_n(\mathbb{R})$ are arbitrary, then $B = \frac{1}{2}CA = \frac{1}{2}C^T A$ satisfies

$$df_A(B) = C,$$

showing that df_A is surjective for all $A \in f^{-1}(I_n)$. It follows that I_n is a regular value of f as desired.

Orientation

Definition 3.2. Let M be a C^k manifold, where $k \geq 1$. We say that M is *orientable* if there is a C^k-atlas $\Phi = \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ representing the C^k-structure on M such that

($\star$) For each $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \varnothing$, the transition function $\phi_\beta \circ \phi_\alpha^{-1}$ satisfies $\det(d(\phi_\beta \circ \phi_\alpha^{-1})_x) > 0$ for each $x \in \phi_\alpha(U_\alpha \cap U_\beta)$.

If M is orientable, an *orientation* of M is a choice of a C^k-atlas satisfying ($\star$). If Φ and Ψ are two C^k-atlases satisfying ($\star$), then they determine the same orientation if their union $\Phi \cup \Psi$ satisfies ($\star$)

Example 3.3. Suppose that $\Phi = \{(U_1, \phi_1), (U_2, \phi_2)\}$ is a C^k-atlas of a C^k-manifold M such that the intersection $U_1 \cap U_2$ is connected. We claim that M is orientable. Indeed, since the determinant of $\det(d(\phi_2 \circ \phi_1^{-1})_x)$ is a continuous map from the connected set $\phi_1(U_1 \cup U_2)$ to $\mathbb{R} \setminus \{0\}$, it is either always positive or always negative on $\phi_1(U_1 \cup U_2)$. If it is always positive then Φ determines an orientation; if it is always negative, then we can change the sign of one of the coordinates of ϕ_2 to make it always positive.

By Assignment 1 (1) and the above observation, S^n is orientable for any $n \geq 2$. It is easy to see that S^1 is also orientable.

Lemma 3.4. *Let $L : \mathbb{C}^n \to \mathbb{C}^n$ be a $\mathbb{C}$-linear isomorphism given by $v \mapsto Cv$ for some complex $n \times n$ matrix $C \in M_n(\mathbb{R})$. Write $C = A + iB$ for some $A, B \in M_n(\mathbb{R})$. Let $i : \mathbb{R}^{2n} \to \mathbb{C}^n$ be the $\mathbb{R}$-linear map given by $(x, y) \mapsto x + iy$. Let $L' : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denote the $\mathbb{R}$-linear map such that $L \circ i = i \circ L'$. Then we see that L' is given by*

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and

$$\det \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = |\det C|^2.$$

Example 3.5. We may form complex projective space $P_n(\mathbb{C})$ in a similar fashion to real projective space. We claim that this $2n$-dimensional manifold is orientable. Indeed, for each $x \in \phi_i(U_i)$, the differential $d(\phi_j \circ \phi_i^{-1})_x : \mathbb{C}^n \to \mathbb{C}^n$ is a $\mathbb{C}$-linear isomorphism. By the Lemma, it follows that if we view the differential as an $\mathbb{R}$-linear map from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$, then it has positive determinant.

This argument shows that a complex n-manifold is an orientable C^∞ $2n$-manifold; indeed, the orientation is determined by the complex structure, so it is an *oriented* C^∞ $2n$-manifold.

Example 3.6. We will see later the real projective space $P_n(\mathbb{R})$ is orientable iff n is odd. In particular, the real projective line $P_1(\mathbb{R}) \cong S^1$ is orientable, and the real projective plane $P_2(\mathbb{R})$ is nonorientable.

Tangent spaces and tangent bundles

Let M be a C^k manifold of dimension n, where $k \geq 1$.

Definition 3.7 (tangent space, tangent vector)**.** Let (U, ϕ) and (V, ψ) be two charts for M around $p \in M$. For vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, we write $(U, \phi, \vec{u}) \sim_p (V, \psi, \vec{v})$ if

$$d(\psi \circ \phi^{-1})_{\phi(p)}(\vec{u}) = \vec{v}.$$

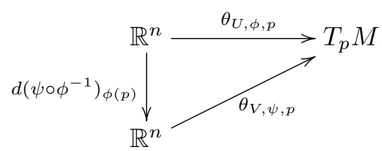
This defines an equivalence relation on such triples, and we let $[(U, \phi, \vec{u})]$ denote the equivalence class of such a triple under this relation. We define the *tangent space to M at p* to be the set

$$T_pM = \{[(U, \phi, \vec{u})] : (U, \phi) \text{ is a chart around } p,\ \vec{u} \in \mathbb{R}^n\}.$$

For a fixed chart (U, ϕ) around p, the map $\theta_{U,\phi,p} : \mathbb{R}^n \to T_pM$ described by

$$\theta_{U,\phi,p}(\vec{u}) = [(U, \phi, \vec{u})]$$

is a bijection (Assignment 3 (1)). This implies that we may endow the space T_pM with an $\mathbb{R}$-linear structure. Moreover, this structure does not depend on the choice of chart: Indeed if (V, ψ) is another chart around p, then the following diagram commutes



and the map $d(\psi \circ \phi^{-1})_{\phi(p)}$ is an $\mathbb{R}$-linear isomorphism.

A *tangent vector at p* is a vector in the n-dimensional real vector space T_pM.

We construct now a $2n$-dimensional manifold called the tangent bundle of M, denoted TM.

1. As a *set*, the tangent bundle of M is given by

$$TM = \{(p, v) : p \in M, v \in T_pM\}.$$

There is a surjective map $\pi : TM \to M$ sending (p, v) to p.

2. *Topology*: For a chart (U, ϕ) for M, let $\tilde{\phi} : \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^n$ be the map described by

$$\tilde{\phi}(p, v) = (\phi(p), \theta^{-1}_{U,\phi,p}(v)).$$

Equip the set TM with the topology such that $\tilde{\phi}$ is a homeomorphism for each chart (U, ϕ). This means that a subset A of TM is open if and only if for each chart (U, ϕ) for M, the set $\tilde{\phi}(\pi^{-1}(U) \cap A)$ is open in $\phi(U) \times \mathbb{R}^n$. With this topology, TM is a topological manifold of dimension $2n$.

It can be shown that that if M is Hausdorff (resp. has a countable basis), then TM is Hausdorff (resp. has a countable basis) as well.

3. *Transition functions*: Note that if U is an open subset of M then $\pi^{-1}(U)$ can be identified with TU. We have $\pi^{-1}(U) \cap \pi^{-1}(V) = TU \cap TV = T(U \cap V) = \pi^{-1}(U \cap V)$. Given two charts (U, ϕ) and (V, ψ) for M, $(TU, \widetilde{\phi})$ and $(TV, \widetilde{\psi})$ are charts for TM, and the transition function

$$\widetilde{\psi} \circ \widetilde{\phi}^{-1} : \tilde{\phi}(TU \cap TV) = \tilde{\phi}(T(U \cap V)) \to \tilde{\psi}(TU \cap TV) = \tilde{\psi}(T(U \cap V))$$

is given by

$$\widetilde{\psi} \circ \widetilde{\phi}^{-1}(\vec{x}, \vec{u}) = (\psi \circ \phi^{-1}(\vec{x}), d(\psi \circ \phi^{-1})_{\vec{x}}(\vec{u}))$$

where $\psi \circ \phi^{-1}(\vec{x})$ is C^k in $\vec{x}$ and the map $\vec{x} \mapsto d(\psi \circ \phi^{-1})_{\vec{x}}$ is C^{k-1} in $\vec{x}$. So $\widetilde{\psi} \circ \widetilde{\phi}^{-1}$ is a C^{k-1} diffeomorphism. It follows that TM is a C^{k-1}-manifold. In particular, if M is a C^∞ manifold then TM is a C^∞ manifold.

Lemma 3.8. *The projection map $\pi : TM \to M$ is a C^{k-1} map. In particular, when $k = \infty$, $\pi : TM \to M$ is a smooth map and a submersion.*

Proof. Given a point (p, v) in TM, where $p \in M$ and $v \in T_pM$, let (U, ϕ) be a C^k chart for M around $p = \pi(p, v)$. Then $(\pi^{-1}(U) = TU, \widetilde{\phi})$ is a C^{k-1} chart around (p, v), and we have the following commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\pi} & U \\ {\scriptstyle \widetilde{\phi}}\downarrow & & \downarrow{\scriptstyle \phi} \\ \phi(U) \times \mathbb{R}^n & \xrightarrow{g} & \phi(U) \end{array}$$

where $g(\vec{x}, \vec{u}) = \vec{x}$ is the restriction of the canonical submersion $\mathbb{R}^{2n} \to \mathbb{R}^n$. □

Assignment 2 (2): TM is orientable, even though M may not be.

4. Monday, September 21, 2015

The differential of a C^k map

Definition 4.1. Let $f : M \to N$ be a C^k map between C^k manifolds of dimension m and n respectively, where $k \geq 1$. The *differential of f at p* is the linear map

$$df_p : T_pM \to T_{f(p)}N$$

defined as follows: Given a chart (U, ϕ) for M around p and a chart (V, ψ) for N around $f(p)$ such that $f(U) \subset V$, let $g := \psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$, and let df_p denote the composition

$$df_p = \theta_{V,\psi,f(p)} \circ dg_{\phi(p)} \circ \theta^{-1}_{U,\phi,p}.$$

In terms of diagrams, this is the map given below

$$\begin{array}{ccc} T_pM & \xrightarrow{df_p} & T_{f(p)}N \\ {\scriptstyle \theta^{-1}_{U,\phi,p}}\downarrow & & \uparrow{\scriptstyle \theta_{V,\psi,f(p)}} \\ \mathbb{R}^m & \xrightarrow{dg_{\phi(p)}} & \mathbb{R}^n \end{array}$$

Remark 4.2. At first glance, it seems that the differential df_p may be ill-defined: a different choice of charts seems to lead to a different definition of df_p. However, the chain rule again comes to our rescue, and one can indeed show that df_p is a well-defined map that is independent of the choice of charts.

Note that df_p is indeed a linear map since the θ and $dg_{\phi(p)}$ are.

Finally, note that this definition is consistent with the case when M is an open subset of $\mathbb{R}^m$ and N is an open subset of $\mathbb{R}^n$.

Theorem 4.3 (Chain Rule)**.** *Let $f : M_1 \to M_2$ and $g : M_2 \to M_3$ be C^k maps between C^k manifolds, where $k \geq 1$. Then*

1. *The composition $g \circ f : M_1 \to M_3$ is a C^k map.*
2. *For each point p in M_1, the differential of the composition is given by*

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

The following definition is equivalent to Definition 2.11 when $k = \infty$.

Definition 4.4. Let $f : M \to N$ be a C^k map between C^kmanifolds, where $k \geq 1$. We say f is a *submersion at p* (resp. *immersion at p*) if df_p is surjective (resp. injective).

Remark 4.5. Suppose that M is a submanifold of N. Then for each p in M, the tangent space T_pM can be viewed as a subspace of T_pN. Indeed, if $i : M \to N$ denotes the inclusion, then $di_p : T_pM \to T_pN$ is injective.

Remark 4.6. Suppose that $f : M \to N$ is a smooth map. Let $q \in N$ be a regular value. By Theorem 2.18 (the preimage theorem), $S = f^{-1}(q)$ is a submanifold of M of dimension $m - n$, where $m = \dim M$ and $n = \dim N$. For each $p \in S$, the tangent space T_pS is given by $T_pS = \ker(df_p : T_pM \to T_{f(p)}N)$. That is, we have the following short exact sequence of real vector spaces

$$0 \longrightarrow T_pS \longrightarrow T_pM \longrightarrow T_qN \longrightarrow 0.$$

Remark 4.7. For every point $p \in \mathbb{R}^n$, we have an isomorphism $T_p\mathbb{R}^n \cong \mathbb{R}^n$ given by $v \mapsto \theta^{-1}_{\mathbb{R}^n,\mathrm{id},p}(v)$. We also have $\tilde{\mathrm{id}} : T\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$.

Example 4.8. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be the map $f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$. We have already seen that 1 is a regular value of f, and thus the unit sphere $S^n = f^{-1}(1)$ is a submanifold of $\mathbb{R}^{n+1}$. For each $p \in S^n$, we compute

$$T_pS^n = \{v \in \mathbb{R}^{n+1} : df_p(v) = 0\} = \{v \in \mathbb{R}^{n+1} : p \cdot v = 0\}$$

Example 4.9. Let $f : M_n(\mathbb{R}) \to S_n(\mathbb{R})$ be the map of Example 3.1, that is, $f(A) = AA^T$. Recall that the orthogonal group $O(n)$ is the preimage of the regular value I_n. For $A \in O(n)$, we compute

$$T_AO(n) = \{B \in M_n(\mathbb{R}) : df_A(B) = 0\} = \{B \in M_n(\mathbb{R}) : BA^T + AB^T = 0\}.$$

In particular, $T_{I_n}O(n) = \{B \in M_n(\mathbb{R}) : B + B^T = 0\} \cong \mathbb{R}^{\frac{n(n-1)}{2}}$ is the set of real $n \times n$ skew-symmetric matrices.

Definition 4.10. Let $f : M \to N$ be a C^k map between C^k manifolds. Define $df : TM \to TN$ by the rule

$$df(p, v) = (f(p), df_p(v)).$$

Proposition 4.11. *Let $f : M \to N$ be a C^k map between C^k manifolds. Then $df : TM \to TN$ is a C^{k-1} map between C^{k-1} manifolds.*

Proposition 4.12. *If M is a smooth submanifold of N of dimension m, then TM is a smooth submanifold of TN of dimension $2m$.*

Example 4.13. The tangent bundle of the sphere S^n is given by

$$TS^n = \{(x, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : |x|^2 = 1, x \cdot v = 0\} \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}.$$

Example 4.14. The tangent bundle of the orthogonal group $O(n)$ is given by

$$TO(n) = \{(A, B) \in M_n(\mathbb{R})\times M_n(\mathbb{R}) : AA^T = I_n, BA^T + AB^T = 0\} \subset M_n(\mathbb{R})\times M_n(\mathbb{R}).$$

Vector bundles

Roughly speaking, a real vector bundle of rank r over a manifold M consists of a family of r-dimensional real vector spaces parametrized by M.

Definition 4.15. Let M be a C^k manifold. A *real C^k vector bundle of rank r over M* consists of

- a C^k manifold E called the *total space* and
- a C^k surjective map $\pi : E \to M$

such that

(i) (local trivialization) There is an open cover $\{U_\alpha : \alpha \in I\}$ of M (where U_α is not necessarily a coordinate neighborhood) and C^k diffeomorphisms $h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$ (called *local trivializations*) such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{h_\alpha} & U_\alpha \times \mathbb{R}^r \\ {\scriptstyle \pi_\alpha}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_1} & \\ U_\alpha & & \end{array}$$

where π_α is the restriction of π to $\pi^{-1}(U_\alpha)$, and pr_1 is the projection to the first factor.

(ii) (transition functions) If the intersection $U_\alpha \cap U_\beta$ is nonempty, then the map

$$h_\beta \circ h_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r$$

is a C^k diffeomorphism of the form $h_\beta \circ h_\alpha^{-1}(x, v) = (x, g_{\beta\alpha}(x)v)$ where $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R})$ is a C^k map. (Note that $GL(r, \mathbb{R}) = \{A \in M_r(\mathbb{R}) : \det(A) \neq 0\}$ is an open subset of $M_r(\mathbb{R}) \cong \mathbb{R}^{r^2}$.)

Remark 4.16. From condition (i), we know that $h_\beta \circ h_\alpha^{-1}$ is a C^k diffeomorphism of the form $(x, v) \mapsto (x, g_{\beta\alpha}(x)v)$ where $g_{\beta\alpha}(x) : \mathbb{R}^r \to \mathbb{R}^r$ is a C^k diffeomorphism (depending on $x \in U_\alpha \cap U_\beta$). However, in condition (ii), we require something stronger: namely that $g_{\beta\alpha}(x)$ is a linear isomorphism. If we only had the weaker condition, then we would say that $\pi : E \to M$ is a *fiber bundle* with total space E and fiber $\mathbb{R}^r$.

Example 4.17 (product vector bundle)**.** The product vector bundle of rank r consists of $\pi = \mathrm{pr}_1 : E = M \times \mathbb{R}^r \to M$ where pr_1 denotes the projection onto the first factor.

Definition 4.18 (trivial vector bundle)**.** We say that $\pi : E \to M$ is a *trivial vector bundle of rank* r if there is a C^k diffeomorphism (when $k \geq 1$) or a homeomorphism (when $k = 0$) $h : E \to M \times \mathbb{R}^r$ such that

- h commutes with the projection maps in the sense that $\pi = \mathrm{pr}_1 \circ h$
- the restriction of h to each fiber $h_x : E_x \to \{x\} \times \mathbb{R}^r$ is a linear isomorphism.

In other words, $\pi : E \to M$ is a trivial vector bundle of rank r if there exists a *global* trivialization $h : E \to M \times \mathbb{R}^r$.

5. Wednesday, September 23, 2015

Vector bundles (continued)

Example 5.1 (tangent bundle)**.** Suppose that M is a C^k manifold with dimension n. Then $\pi : TM \to M$ is a C^{k-1} vector bundle of rank n over M.

To see this, let $\Phi = \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ be a C^k-atlas of the C^k manifold M, define local trivializations $h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ by

$$h_\alpha(p, v) = \left(p, \theta^{-1}_{U_\alpha, \phi_\alpha, p}(v)\right)$$

where $p \in U_\alpha$ and $v \in T_pM$. Then each h_α is C^{k-1} diffeomorphism which satisfies (i) in Definition 4.15. If $U_\alpha \cap U_\beta \neq \emptyset$, the transition function

$$h_\beta \circ h_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is given by

$$h_\beta \circ h_\alpha^{-1}(p, \vec{u}) = \big(p, d(\phi_\beta \circ \phi_\alpha^{-1})_{\phi_\alpha(p)}(\vec{u})\big).$$

Note that $p \mapsto d(\phi_\beta \circ \phi_\alpha^{-1})_{\phi_\alpha(p)}$ defines a C^{k-1} map from $U_\alpha \cap U_\beta$ to $GL(n, \mathbb{R})$. So the transition functions satisfy (ii) in Definition 4.15.

Example 5.2 (universal line bundle over $P_n(\mathbb{R})$)**.** See Assignment 3 (2).

Definition 5.3. Let $\pi : E \to M$ be a C^k vector bundle over a C^k manifold M. A C^k *section* of π is a C^k map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$.

Lemma 5.4. *Let $\pi : E \to M$ be a C^k vector bundle of rank r over a C^k manifold M. Then $\pi : E \to M$ is trivial if and only if there are C^k sections $s_1, \dots, s_r$ of $\pi : E \to M$ such that for each point $x \in M$, the collection $\{s_1(x), \dots, s_r(x)\}$ forms a basis of E_x.*

Proof. ($\Rightarrow$) Suppose that $\pi : E \to M$ is trivial and let $h : E \to M \times \mathbb{R}^r$ be a trivialization as in Definition 4.18. Let $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_r = (0, \dots, 0, 1)$ be the standard basis of $\mathbb{R}^r$. Define $s_i : M \to E$ by $s_i(x) = h^{-1}(x, e_i)$, $i = 1, \dots, r$. Then s_i are C^k sections of $\pi : E \to M$, and for each $x \in M$ the collection $\{s_1(x), \dots, s_r(x)\}$ forms a basis of $E_x \cong \mathbb{R}^r$.
($\Leftarrow$) Conversely, if we are given C^k sections $s_1, \dots, s_r$ of $\pi : E \to M$ such that the collection $\{s_1(x), \dots, s_r(x)\}$ forms a basis of $E_x \cong \mathbb{R}^r$ for all $x \in M$, we define $\psi : M \times \mathbb{R}^r \to E$ by

$$(x, (v_1, \dots, v_r)) \mapsto (x, \sum_{i=1}^{r} v_i s_i(x)).$$

where $x \in M$, $(v_1, \dots, v_r) \in \mathbb{R}^r$, and $\sum_{i=1}^r v_i s_i(x) \in E_x$. Then ψ is a C^k-diffeomorphism (when $k \geq 1$) or a homeomorphism (when $k = 0$), and $h := \psi^{-1} : E \to M \times \mathbb{R}^r$ is a global trivialization as in Definition 4.18. □

Definition 5.5. Let M be a smooth manifold. A *smooth vector field* on M is a smooth section of TM.

Derivations

Definition 5.6. Let M be a C^k manifold and let p be a point of M. Let U and V be open neighborhoods of p in M and let $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ be C^k functions. We define an equivalence relation $\sim_p$ by the rule $(f : U \to \mathbb{R}) \sim_p (g : V \to \mathbb{R})$ if and only if there is an open neighborhood W of p such that $W \subset U \cap V$ and $f|_W \equiv g|_W$.

A *germ of C^k functions at p* is an equivalence class under this equivalence relation. Let $[f : U \to \mathbb{R}]$ denote the equivalence class represented by $f : U \to \mathbb{R}$. We let $C^k_p(M)$ denote the collection of all such equivalence classes:

$$C^k_p(M) := \{(f : U \to \mathbb{R}) : U \text{ is an open neighborhood of } p \text{ in } M,\ f \text{ is a } C^k \text{ function on } U\}/\sim_p .$$

Lemma 5.7. *The set $C^k_p(M)$ of germs of C^k-functions at p has the natural structure of a ring:*

$$\begin{aligned} [f : U \to \mathbb{R}] + [g : V \to \mathbb{R}] &= [f + g : U \cap V \to \mathbb{R}], \\ [f : U \to \mathbb{R}] \cdot [g : V \to \mathbb{R}] &= [f \cdot g : U \cap V \to \mathbb{R}], \end{aligned}$$

where $(f + g)(q) = f(q) + g(q)$ and $(f \cdot g)(q) = f(q)g(q)$ for $q \in U \cap V$.

Remark 5.8. In the definition of $C_p^k(M)$ in Definition 5.6, we may assume that U is contained in some fixed coordinate chart (U_0, ϕ_0) for M around p, and hence we get a map

$$C_p^k(M) \to C_0^k(\mathbb{R}^n)$$
$$[f : U \to \mathbb{R}] \mapsto [f \circ \phi_0^{-1} : \phi_0(U) \to \mathbb{R}].$$

which is a ring isomorphism. Therefore, it is sufficient to study germs of C^k functions at 0 in $\mathbb{R}^n$.

Lemma 5.9. *Let $C^k(M)$ be the set of all C^k-functions on M. The natural map $C^k(M) \to C_p^k(M)$ given by $f \mapsto [f : M \to \mathbb{R}]$ is surjective.*

Proof. Suppose we have a C^k function $f : U \to \mathbb{R}$ defined on a open neighborhood U of p. We claim that there is a neighborhood U' containing p and a C^k-map $\beta : U' \to \mathbb{R}$ such that

- $U' \subset U$
- $\overline{U'}$ is compact
- $\beta(x) = 1$ for each $x \in U'$
- $\mathrm{supp}(\beta)$ is relatively compact in U
- $\beta(x) = 0$ for all $x \notin U$.

Then the multiplication $(\beta f : U \to \mathbb{R}) \sim_p (f : U \to \mathbb{R})$. But βf extends to a C^k function defined on all of M. The result now follows. □

Definition 5.10. A *derivation* on $C_p^k(M)$ is an $\mathbb{R}$-linear map $\delta : C_p^k(M) \to \mathbb{R}$ such that

$$\delta(fg) = \delta(f)g(p) + f(p)\delta(g) \qquad \text{(Leibniz rule)}$$

for each $f, g \in C_p^k(M)$.

Remark 5.11. The set of derivations on $C_p^k(M)$ is an $\mathbb{R}$-linear space.

Example 5.12. Suppose that $k \geq 1$. For $i = 1, \dots, n$,

$$\frac{\partial}{\partial x_i}(0) : C_0^k(\mathbb{R}^n) \to \mathbb{R}, \quad f \mapsto \frac{\partial f}{\partial x_i}(0).$$

is a derivation on $C_0^k(\mathbb{R}^n)$. For any $a_1, \dots, a_n \in \mathbb{R}$,

$$\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}(0) : C_0^k(\mathbb{R}^n) \to \mathbb{R}, \quad f \mapsto \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}(0)$$

is a derivation on $C_0^k(\mathbb{R})$.

Lemma 5.13. *This lemma has three parts.*

(a) *If δ is a derivation on $C_0^k(\mathbb{R}^n)$ and f is constant near 0, then $\delta(f) = 0$.*
(b) *If δ is a derivation on $C_0^0(\mathbb{R}^n)$, then $\delta \equiv 0$.*
(c) *If δ is a derivation on $C_0^\infty(\mathbb{R}^n)$, then we may write*

$$\delta = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}(0)$$

where $a_i = \delta(x_i)$.

Proof. (a) Since δ is linear, it suffices to show that $\delta(1) = 0$, but this is indeed the case as

$$\delta(1) = \delta(1 \cdot 1) = \delta(1)1 + 1\delta(1) = 2\delta(1).$$

(b) Assignment 3 (3).
(c) Let f be a smooth function on $\mathbb{R}^n$ defined on a neighborhood of 0. Take x small enough such that the map $g : (-2, 2) \to \mathbb{R}$ defined by $g(t) = f(tx)$ is defined. Then $g(t)$ is a smooth function on $(-2, 2)$.

$$f(x)-f(0) = g(1)-g(0) = \int_0^1 g'(t)dt = \int_0^1 \Big(\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(tx)\Big)dt = \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt$$

Let $h_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt$. Then $h_i \in C_0^\infty(\mathbb{R}^n)$ and

$$h_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(0)dt = \frac{\partial f}{\partial x_i}(0).$$

It then follows that

$$\delta(f) = \delta(f-f(0)) = \delta\Big(\sum_{i=1}^n x_i h_i(x)\Big) = \sum_{i=1}^n (\delta(x_i)h_i(0)+x_i(0)\delta(h_i)) = \sum_{i=1}^n \delta(x_i)\frac{\partial f}{\partial x_i}(0)$$

as desired. □

Let D_pM denote the space of derivations on $C_p^\infty(M)$. We claim that there is a linear isomorphism

$$T_pM \longrightarrow D_pM$$
$$[(U, \phi, \vec{u})] \mapsto \sum_{i=1}^n u_i \frac{\partial}{\partial x_i}(p).$$

where the derivation $\frac{\partial}{\partial x_i}(p) : C_p^\infty(M) \to \mathbb{R}$ is defined by $f \mapsto \frac{\partial(f\circ\phi^{-1})}{\partial x_i}(\phi(p))$. Indeed, if this is well-defined, it is clearly a linear isomorphism, so it suffices to show that it is well-defined.

Let (V, ψ) be another chart for M around p. Let $v \in \mathbb{R}^n$ be such that $[(U, \phi, \vec{u})] = [(V, \psi, \vec{v})]$. Then this means that $\vec{v} = d(\psi \circ \phi^{-1})_{\phi(p)}(\vec{u})$. Write $\phi = (x_1, \dots, x_n)$ and $\psi = (y_1, \dots, y_n)$. Then the fact that $\vec{v} = d(\psi \circ \phi^{-1})_{\phi(p)}\vec{u}$ implies that

$$v_j = \sum_{i=1}^n \frac{\partial y_j}{\partial x_i}(\phi(p))u_i,$$

We then apply the chain rule to see that

$$\sum_{i=1}^n u_i \frac{\partial}{\partial x_i}(p) = \sum_{i,j=1}^n u_i \frac{\partial y_j}{\partial x_i}(\phi(p))\frac{\partial}{\partial y_j}(p) = \sum_{j=1}^n v_j \frac{\partial}{\partial y_j}(p).$$

$\sum_{i=1}^n u_i \frac{\partial}{\partial x_i}(p)$ is the notation of a tangent vector at $p \in M$ in do Carmo's book.

Let (U, ϕ) be a coordinate chart for M and write $\phi = (x_1, \dots, x_n)$. Recall that $\widetilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n$ is defined by $\widetilde{\phi}(p, v) = (\phi(p), \theta_{U,\phi,p}^{-1}(v))$, and the linear isomorphism $T_pM \xrightarrow{\cong} D_pM$ is given by $\theta_{U,\phi,p}(\vec{u}) \mapsto \sum_{i=1}^n u_i \frac{\partial}{\partial x_i}(p)$. So $\widetilde{\phi}^{-1}$: $\phi(U) \times \mathbb{R}^n \to TU$ is given by

$$\widetilde{\phi}^{-1}(x, \vec{u}) = (\phi^{-1}(x), \sum_{i=1}^n u_i \frac{\partial}{\partial x_i}(p))$$

where $x \in \phi(U) \subset \mathbb{R}^n$ and $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$. For $i = 1, \ldots, n$

$$\frac{\partial}{\partial x_i} : U \to TU, \quad p \mapsto (p, \frac{\partial}{\partial x_i}(p))$$

are smooth sections of $TU \to U$. Moreover, for each point $p \in U$, the collection $\{\frac{\partial}{\partial x_i}(p) : i = 1 \ldots, n\}$ forms a basis for T_pU, and hence the collection $\{\frac{\partial}{\partial x_i} : i = 1, \ldots, n\}$ forms a C^∞ frame for $TU \to U$. We let $C^\infty(U, TU)$ denote the space of C^∞ sections of $TU \to U$. We have an isomorphism

$$T_pM = \bigoplus_{i=1}^n \mathbb{R} \frac{\partial}{\partial x_i}(p)$$

as real vector spaces, and an isomorphism

$$C^\infty(U, TU) = \bigoplus_{i=1}^n C^\infty(U) \frac{\partial}{\partial x_i}$$

as $C^\infty(U)$-modules. Therefore, any C^∞ vector field on U is of the form

$$\sum_i a_i \frac{\partial}{\partial x_i}, \quad a_i \in C^\infty(U).$$

6. Monday, September 28, 2015

Lie derivative and Lie bracket

Last time we defined derivations on the germs of smooth functions of M at p. We also identified the set of derivations D_pM with the tangent space T_pM.

Definition 6.1. Let M be a smooth manifold. A *derivation* on $C^\infty(M)$ is an $\mathbb{R}$-linear map $\delta : C^\infty(M) \to C^\infty(M)$ satisfying the Leibniz rule

$$\delta(fg) = \delta(f)g + f\delta(g).$$

Let $D(M)$ denote the set of derivations on $C^\infty(M)$.

Remark 6.2. This is a sort of global extension of the previous definition.

Remark 6.3. Note that $D(M)$ is a $C^\infty(M)$-module: Indeed if $\delta \in D(M)$ and $h \in C^\infty(M)$, then we can define $h\delta \in D(M)$ by the rule

$$(h\delta)(f) = h\delta(f).$$

Now we relate this notion to vector fields, via Lie derivatives.

Definition 6.4. Let X be a smooth vector field on a smooth manifold M. Define a map $L_X : C^\infty(M) \to C^\infty(M)$ called the *Lie derivative* by the rule

$$(L_X f)(p) = X(p)f$$

for any $p \in M$. Recall that a smooth vector field is a smooth section $M \to TM$, so that means that $X(p) \in T_pM = D_pM$, so we may apply $X(p)$ to the germ determined by f at p. We sometimes denote $L_X f$ by Xf.

To see Xf is a smooth function on a coordinate neighborhood U of p, recall that X restricted to U is given by $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ where $a_i \in C^\infty(U)$. Then we see that

$$(Xf)(p) = \sum_{i=1}^n a_i(p) \frac{\partial}{\partial x_i}(f \circ \phi^{-1})(\phi(p)).$$

In do Carmo's notation, we write

$$(Xf)(p) = \sum_{i=1}^n a_i(p) \frac{\partial f}{\partial x_i}(p).$$

Theorem 6.5. *The assignment*

$$C^\infty(M, TM) \to D(M)$$
$$X \mapsto L_X$$

is an isomorphism of $C^\infty(M)$-modules.

Proof. We provide an outline of the proof. First it is clear that the map is $C^\infty(M)$-linear.

To see that the map is surjective, suppose we are given $\delta \in D(M)$, we will define $X \in C^\infty(M, TM)$ such that $L_X = \delta$. For any $p \in M$, we let (U, ϕ) be a coordinate chart for M around p and we let $X(p) = \sum_{i=1}^n \delta_p(x_i) \frac{\partial}{\partial x_i}(p)$. Here the notation δ_p means that we restrict the derivation δ to the germs of functions at p.

To see that the map is injective, we want to show that if $X \in C^\infty(M, TM)$ is not identically zero, then L_X is not identically zero. If $X \neq 0$, then there is a point $p \in M$ such that $X(p) \neq 0 \in T_pM = D_pM$. So there is an $f \in C_p^\infty(M)$ such that $X(p)f \neq 0$. We may assume that $f \in C^\infty(M)$. Then $(L_X f)(p) = X(p)f \neq 0$. □

Definition 6.6 (Lie bracket)**.** Let X, Y be smooth vector fields on M. We define $[X, Y] : C^\infty(M) \to C^\infty(M)$ by the rule

$$[X, Y](f) = XYf - YXf = L_X L_Y f - L_Y L_X f.$$

Lemma 6.7. *The map $[X, Y]$ is a derivation.*

Proof. It is clear that $[X, Y]$ is $\mathbb{R}$-linear. We need to check the Leibniz rule. But this is straightforward and left as an exercise. □

By the Lemma and the Theorem, we may view $[X, Y]$ as a smooth vector field. In local coordinates (U, ϕ), suppose that $X = \sum_i a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_i b_i \frac{\partial}{\partial x_i}$. Then in terms of local coordinates we find that

$$[X, Y] = \sum_{i,j} \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

Proposition 6.8. *The map $[-, -] : C^\infty(M, TM) \times C^\infty(M, TM) \to C^\infty(M, TM)$ defines a map which satisfies the following properties:*

(i) $[-, -]$ *is $\mathbb{R}$-bilinear.*
(ii) $[-, -]$ *is anti-commutative in the sense that $[X, Y] = -[Y, X]$.*
(iii) $[-, -]$ *satisfies the Jacobi identity in the sense that*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

(iv) *If $f, g \in C^\infty(M)$, then*

$$[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X.$$

Remark 6.9. The first three properties show that $(C^\infty(M, TM), [-,-])$ is a Lie algebra over $\mathbb{R}$.

Proof of Proposition 6.8. (i) and (ii) are clear from definition. It is straightforward to check (iii) and (iv); you will be asked to verify (iii) in Assignment 4 (1). □

We now discuss the differential in terms of derivations.

Definition 6.10. Let $F: M \to N$ be a C^k map between C^k manifolds. Let l be a positive integer satisfying $l \le k$. Then F induces a map $F^*: C^l(N) \to C^l(M)$ called the *pullback* defined by the rule $f \mapsto f \circ F$. If p is a point in M, we get a map $F_p^*: C^l_{F(p)}(N) \to C^l_p(M)$ defined by $[(V, f)] \mapsto [(F^{-1}(V), f \circ F)]$.

Remark 6.11. If M and N are C^k manifolds and $F: M \to N$ is a continuous map, then for each $p \in M$, we get a map $F_p^*: C^0_{F(p)}(N) \to C^0_p(M)$. Then F is a C^k map if and only if for each p in M, the image $F_p^*(C^k_{F(p)}(N))$ is a subring of $C^k_p(M)$. We may use this to define C^k maps. (cf. Roundtable on September 25, and Well's *Differential Analysis on Complex Manifolds*, Chapter I)

Lemma 6.12. *Let $F: M \to N$ be a smooth map between smooth manifolds. For each point p in M, the map $dF_p: T_pM = D_pM \to T_{F(p)}N = D_{F(p)}N$ is given by*

$$dF_p(X)f = X(F^*f) \tag{6.1}$$

for any $X \in T_pM = D_pM$ and $f \in C^\infty_{F(p)}(N)$.

Proof. This follows from the chain rule. Passing to local coordinates, we may assume that M is an open subset of $\mathbb{R}^m$, N is an open subset of $\mathbb{R}^n$, $p = 0$, and $F(p) = 0$. We write $F(x) = (y_1(x), \ldots, y_n(x))$. Then any derivation $X \in D_0\mathbb{R}^m$ is given by $X = \sum_{i=1}^m a_i \frac{\partial}{\partial x_i}(0)$. Note that

$$dF_p(X) = \sum_{j=1}^n \left(\sum_{i=1}^m \frac{\partial y_j}{\partial x_i}(0) a_i \right) \frac{\partial}{\partial y_j}(0).$$

The LHS and RHS of (6.1) are

$$\text{LHS} = dF_p(X)f = \sum_{i=1}^m \sum_{j=1}^n a_i \frac{\partial y_j}{\partial x_i}(0) \frac{\partial f}{\partial y_j}(0), \qquad \text{RHS} = \sum_{i=1}^m a_i \frac{\partial}{\partial x_i}(f \circ F)(0).$$

which are equal by the chain rule. □

Remark 6.13. We may use (6.1) to *define* dF_p.

Definition 6.14. Let M be a smooth manifold. A *smooth curve in M* is a smooth map $\gamma: (a, b) \to M$ where $-\infty \le a < b \le +\infty$.

Notation 6.15. For any $t \in (a, b)$, let $\gamma'(t)$ (or $\frac{d\gamma}{dt}(t)$) denote the tangent vector $d\gamma_t\left(\frac{\partial}{\partial t}\right) \in T_{\gamma(t)}M$.

Example 6.16. If $M = \mathbb{R}^n$, then a smooth map $\gamma: (a, b) \to M$ is given by

$$\gamma(t) = (x_1(t), \ldots x_n(t))$$

where $x_i: (a, b) \to \mathbb{R}$ are C^∞ function on (a, b).

$$\gamma'(t) = (x_1'(t), \ldots, x_n'(t)) = \sum_{i=1}^n x_i'(t) \frac{\partial}{\partial x_i}(\gamma(t)).$$

Lemma 6.17. *Let M be a smooth manifold and let $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth curve. Let $\gamma(0) = p$. Then $\gamma'(0)$ is a derivation at p given by*

$$\gamma'(0)f = \frac{d}{dt}(f \circ \gamma)|_{t=0} \tag{6.2}$$

Proof. This is a special case of Lemma 6.12. □

Remark 6.18. do Carmo uses (6.2) to *define* a derivation $\gamma'(0) : C_p^\infty(M) \to \mathbb{R}$ for each smooth curve passing through $p \in M$ at $t = 0$. The tangent space T_pM is *defined* to be the collection of such $\gamma'(0)$. Under this definition, the differential $dF_p : T_pM \to T_{F(p)}N$ of a smooth map $F : M \to N$ at $p \in M$ is defined by

$$\gamma'(0) \mapsto (F \circ \gamma)'(0).$$

7. Wednesday, September 30, 2015

Integral Curves

Definition 7.1. Let X be a smooth vector field on a smooth manifold M and let $\gamma : I \to M$ be a smooth curve. We say that γ is an *integral curve* of X if $\gamma'(t) = X(\gamma(t))$ for all $t \in I$.

Example 7.2. $M = \mathbb{R}^n$

$$\gamma(t) = (x_1(t), \ldots, x_n(t))$$

where $x_i : I \to \mathbb{R}$ are smooth functions on I. A smooth vector field on $\mathbb{R}^n$ is of the form

$$X(x) = (a_1(x), \ldots, a_n(x)) = \sum_i a_i(x)\frac{\partial}{\partial x_i}$$

where a_i are smooth functions on $\mathbb{R}^n$, so X can be viewed as a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^n$. The statement that γ is an integral curve of X is equivalent to a system of ODE's given by

$$\frac{dx_i}{dt}(t) = a_i(x_1(t), \ldots, x_n(t)), \quad i = 1, \ldots, n.$$

Theorem 7.3. *Let M be a smooth manifold and let X be a smooth vector field on M.*

(i) *For any point $p \in M$, there is an open interval $I_p \subset \mathbb{R}$ containing 0 and an integral curve $\phi_p : I_p \to M$ of X such that $\phi_p(0) = p$ and I_p is a maximal interval for such a ϕ_p. Moreover, this integral curve is unique in the following sense. If $\gamma : I' \to M$ is an integral curve of X on an open interval I' containing 0 such that $\gamma(0) = p$, then $I' \subset I_p$ and $\gamma = \phi_p|_{I'}$.*

(ii) *For any $p \in M$ there is*
- *an open neighborhood U of p in M*
- *an open interval I of 0 in $\mathbb{R}$*
- *a smooth map $\phi : I \times U \to M$*

such that

$$\begin{cases} \frac{\partial \phi}{\partial t}(t, q) = & X(\phi(t, q)) \\ \phi(0, q) = & q \end{cases}$$

Proof. We may assume $M = \mathbb{R}^n$ and $p = 0$. Then the proof becomes one in ODE's. Reference: Boothby Chapter IV. □

Example 7.4. If $M = \mathbb{R}^n$ and $p = (a_1, \ldots, a_n)$. Suppose that X is the identity vector field, i.e. $X(\vec{x}) = \vec{x}$ for all $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then the integral curves are straight lines emanating from the origin. In terms of local coordinates,

$$\begin{cases} \frac{dx_i}{dt} &= x_i \\ x_i(0) &= a_i \end{cases} \quad i = 1, \ldots, n,$$

which implies $x_i(t) = a_i e^t$. And $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is given by $\phi(t, x_1, \ldots, x_n) = (x_1 e^t, \ldots, x_n e^t)$, or equivalently, $\phi(t, \vec{x}) = e^t \vec{x}$.

Example 7.5. Let $M = \{\vec{x} \in \mathbb{R}^n : |\vec{x}| < 1\}$ and again X is the identity vector field. If $p = \vec{a} = (a_1, \ldots, a_n)$ then $\phi_p : I_p \to \mathbb{R}^n$ is given by $\phi_p(t) = e^t \vec{a}$, where $I_p = (-\infty, -\log |\vec{a}|)$.

Remark 7.6. If $q = \phi_p(t_0)$, then $\phi_q(t) = \phi_p(t + t_0)$.

Now we change our point of view. Instead of fixing p, we fix time t in the function $\phi(t, p)$. Define $\phi_t : U \to M$ by the rule $\phi_t(q) = \phi(t, q)$. We should think of this as telling us where points in M get mapped after flowing a certain time t. Because of this, we call ϕ_t the *local flow* of X.

Remark 7.7. By the previous remark (Remark 7.6), we find that $\phi_{t_1} \circ \phi_{t_2} = \phi_{t_1 + t_2}$ when both hand sides of the equality are defined.

Lemma 7.8. *Let X be a smooth vector field on a smooth manifold M such that the support of X is compact. Recall that the support of X is*

$$\operatorname{Supp}(X) = \overline{\{p \in M : X(p) \neq 0\}}.$$

Then there exists a unique smooth map $\phi : \mathbb{R} \times M \to M$ such that

$$\frac{\partial \phi}{\partial t}(t, q) = X(\phi(t, q)), \quad \phi(0, q) = q. \tag{7.1}$$

(In other words, we have a global *flow $\phi_t : M \to M$ which exists for all time $t \in \mathbb{R}$.)*

Proof. It suffices to prove the existence; the uniqueness follows from part (i) of Theorem 7.3. Let $K = \operatorname{Supp}(X)$.
1. The set $V := M \setminus K$ is open, and $X(q) = 0$ for $q \in V$. Define $\phi : \mathbb{R} \times V \to M$ by $\phi(t, q) = q$. Then ϕ is smooth, and it satisfies

$$\frac{\partial \phi}{\partial t}(t, q) = 0 = X(q) = X(\phi(t, q)), \quad \phi(0, q) = q.$$

2. Given any $p \in K$, by Theorem 7.3 (ii), there exists an open neighbhood U_p of p in M and a positive number $\epsilon_p > 0$ such that there is a C^∞ map $\psi_p : (-\epsilon_p, \epsilon_p) \times U_p \to M$ satisfying

$$\frac{\partial \psi_p}{\partial t}(t, q) = X(\psi_p(t, q)), \quad \psi_p(0, q) = q.$$

Moreover, if $p_1, p_2 \in K$ and $U_{p_1} \cap U_{p_2} \neq \emptyset$ then part (i) of Theorem 7.3 implies

$$\psi_{p_1}|_{(-\epsilon, \epsilon) \times (U_{p_1} \cap U_{p_2})} = \psi_{p_2}|_{(-\epsilon, \epsilon) \times (U_{p_1} \cap U_{p_2})}$$

where $\epsilon = \min\{\epsilon_{p_1}, \epsilon_{p_2}\} > 0$. So we obtain a smooth map $\psi(t, q)$ defined on $(-\epsilon, \epsilon) \times (U_{p_1} \cup U_{p_2})$.

K is compact and $K \subset \cup_{p\in K} U_p$, so there are finitely many $p_1, \dots, p_N \in K$ such that $K \subset \cup_{i=1}^N U_{p_i}$. Let $\epsilon := \min\{\epsilon_{p_1}, \dots, \epsilon_{p_N}\} > 0$, and define $U := \cup_{i=1}^N U_{p_i}$. Then we obtain a smooth map $\psi : (-\epsilon, \epsilon) \times U \to M$ satisfying

$$\frac{\partial \psi}{\partial t}(t, q) = X(\psi(t, q)), \quad \psi(0, q) = q.$$

3. By part (i) of Theorem 7.3, $\phi|_{(-\epsilon,\epsilon)\times(U\cap V)} = \psi|_{(-\epsilon,\epsilon)\times(U\cap V)}$, where $\phi : \mathbb{R} \times V \to M$ is defined in Step 1 above and $\psi : (-\epsilon, \epsilon) \times U \to M$ is defined in Step 2 above. We also have $U \cup V = M$, so we obtain a smooth map $\phi : (-\epsilon, \epsilon) \times M \to M$ satisfying (7.1).
4. For any $t \in \mathbb{R}$, there exists a positive integer n such that $|t| < n\epsilon$. Define

$$\phi(t, q) := \underbrace{\phi(\frac{t}{n}, \phi(\frac{t}{n}, \cdots \phi(\frac{t}{n},}_{n \text{ times}}\ q\ \underbrace{)) \cdots)}_{n \text{ times}}$$

where $q \in M$; the definition is independent of choice of $n > |t|$. Then $\phi : \mathbb{R} \times M \to M$ is a smooth map satisfying (7.1). $\square$

If ϕ_t is defined on all of M and for all $t \in \mathbb{R}$, then we have a group homomorphism $(\mathbb{R}, +) \to (\mathrm{Diff}(M), \circ)$ defined by $t \mapsto \phi_t$. In particular, ϕ_0 is the identity map. The inverse of ϕ_t is the map ϕ_{-t}. The image of this group homomorphism lies in the connected component of the identity diffeomorphism, since $\mathbb{R}$ is connected.

Flow and Lie derivative

Let M be a smooth manifold and let X be a smooth vector field. We have defined the Lie derivative of X by the rule $L_X(f)(p) = X(p)(f)$. Recall that $L_X : C^\infty(M) \to C^\infty(M)$ is $\mathbb{R}$-linear and satisfies the Leibniz rule. Now we want to extend L_X to a map $L_X : C^\infty(M, TM) \to C^\infty(M, TM)$.

Definition 7.9. We define $L_X : C^\infty(M, TM) \to C^\infty(M, TM)$ by the rule

$$L_X(Y) = [X, Y].$$

Then L_X is an $\mathbb{R}$-linear map. Moreover, it satisfies the following Leibniz rule:

$$L_X(fY) = L_X(f)Y + f(L_X(Y))$$

for any smooth function f and any vector fields Y on M.

Remark 7.10. We have a few remarks.

- If we consider $L_X : C^\infty(M) \to C^\infty(M)$, then we can see that $L_{fX} = fL_X$ if $f \in C^\infty(M)$ and $X \in C^\infty(M, TM)$. So the operator L_X on $C^\infty(M)$ is $C^\infty(M)$-linear in X.
- If we consider $L_X : C^\infty(M, TM) \to C^\infty(M, TM)$, then we can see that

  $$L_{fX}(Y) = [fX, Y] = f[X, Y] - Y(f)X = fL_X(Y) - Y(f)X.$$

  So the operator L_X on $C^\infty(M, TM)$ is $\mathbb{R}$-linear but not $C^\infty(M)$-linear in X.

We now discuss the pushforward and pullback of a vector field under a diffeomorphism.

Definition 7.11. Let $F : M \to N$ be a smooth diffeomorphism and let X be a smooth vector field on M. Then we define the *pushforward* F_*X to be the smooth vector field on N defined by

$$F_*X(p) = (dF)_{F^{-1}(p)}(X(F^{-1}(p))).$$

Given a smooth vector field Y on N, we define the *pullback* of Y to be $F^*Y = (F^{-1})_*(Y)$, which is a smooth vector field on M.

Proposition 7.12. *Let X be a smooth vector field on a smooth manifold M. Let p be a point of M. By Theorem 7.3 (ii), there is an open neighborhood U of p in M and a local flow $\phi_t : U \to M$ of X for t in some small neighborhood $(-\epsilon, \epsilon)$ of 0. Then*

(a) *For each $f \in C_p^\infty(M)$, we compute that*

$$(L_X f)(p) = \frac{d}{dt}\Big|_{t=0} (\phi_t^* f)(p) = \frac{d}{dt}\Big|_{t=0} (f \circ \phi_t)(p).$$

(b) *For a smooth vector field Y defined an on open neighborhood V of p in U, we compute that*

$$(L_X Y)(p) = -\frac{d}{dt}\Big|_{t=0} ((\phi_t)_* Y)(p) = \lim_{t \to 0} \frac{Y(p) - ((\phi_t)_* Y)(p)}{t}.$$

Proof. (a) We compute

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} (f \circ \phi_t)(p) =& \frac{d}{dt}\Big|_{t=0} f(\phi_t(p)) = \frac{d}{dt}\Big|_{t=0} f(\phi_p(t)) \\ =& \frac{d}{dt}\Big|_{t=0} (f \circ \phi_p)(t) = \phi_p'(0) f = X(p) f. \end{aligned}$$

(b) It suffices to show that for any $f \in C_p^\infty(M)$, we have

$$[X, Y](p) f = -\frac{d}{dt}\Big|_{t=0} ((\phi_t)_* Y)(p) f.$$

We then compute

$$((\phi_t)_* Y)(p) f = (d\phi_t)_{\phi_{-t}(p)}(Y(\phi_{-t}(p))) f = Y(\phi_{-t}(p))(f \circ \phi_t),$$

where the second equality follows from Lemma 6.12. Let $h(t, q) = f \circ \phi_t(q) - f(q)$. Then note that h is a smooth map from $(-\delta, \delta) \times V \to \mathbb{R}$ for some small δ and some open neighborhood V of p in M. Then $h(0, q) = 0$ for all $q \in V$. By Lemma 7.13 below, we may write

$$h(t, q) = t g(t, q)$$

where $g : (-\delta, \delta) \times V \to \mathbb{R}$ is some smooth function. Define $g_t : V \to \mathbb{R}$ by the rule $g_t(q) = g(t, q)$. Then $g_t \in C^\infty(V)$. By part (a),

$$(L_X f)(q) = \frac{d}{dt}\Big|_{t=0} (f \circ \phi_t)(q) = \lim_{t \to 0} \frac{f \circ \phi_t(q) - f(q)}{t} = \lim_{t \to 0} g(t, q) = g(0, q) = g_0(q).$$

It follows that $g_0 = Xf \in C_p^\infty(M)$. Then we find that

$$Y(\phi_{-t}(p))(f \circ \phi_t) = Y(\phi_{-t}(p))(f + t g_t) = Y(\phi_{-t}(p))(f) + t Y(\phi_{-t}(p))(g_t).$$

Let $r(t) = Y(\phi_{-t}(p))(g_t)$, which is a smooth function in one variable t. Then we find

$$Y(\phi_{-t}(p))(f \circ \phi_t) = (Yf)(\phi_{-t}(p)) + t \cdot r(t).$$

We now differentiate to find

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}((\phi_t)_*Y)(p)f =& \frac{d}{dt}\Big|_{t=0}(Yf)\circ\phi_{-t}(p) + r(0) = -X(p)(Yf) + Y(p)g_0 \\ =& -X(p)(Yf) + Y(p)(Xf) = -[X,Y](p)f\end{aligned}$$

as desired. □

Lemma 7.13. *Let δ be a small positive number, let U be an open subset of M, and let $h : (-\delta,\delta)\times U \to \mathbb{R}$ be smooth. Suppose that $h(0,q) = 0$ for any $q \in U$. Then $h(t,q) = tg(t,q)$ for some smooth function $g : (-\delta,\delta)\times U \to \mathbb{R}$.*

Proof. Fix t, q. Let $u(s) = h(st,q)$. Then $u(s)$ is C^∞ function of one variable s.

$$\begin{aligned}h(t,q) = h(t,q) - h(0,q) = u(1) - u(0) = \int_0^1 u'(s)ds = \int_0^1 t\frac{\partial h}{\partial t}(st,q)ds \\ = t\int_0^1 \frac{\partial h}{\partial t}(st,q)ds = tg(t,q).\end{aligned}$$

where

$$g(t,q) := \int_0^1 \frac{\partial h}{\partial t}(st,q)ds$$

is a C^∞ function in (t,q) since h is. □

8. Monday, October 5, 2015

Definition 8.1 (Subbundle)**.** Let $\pi : E \to M$ be a smooth vector bundle of rank r over M. A subset F of E is called a *smooth subbundle of rank k* if for any $p \in M$, there is an open neighborhood U of p in M and a local trivialization $h : \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that $h\left(F \cap \pi^{-1}(U)\right) = U \cap (\mathbb{R}^k \times \{0\})$.

Remark 8.2. We have some remarks.

(i) For any $p \in M$, the fiber $F_p = F \cap E_p$ is a k-dimensional subspace of E_p. Moreover, F_p depends smoothly on the choice of p.

(ii) The map $\pi|_F : F \to M$ is a smooth vector bundle of rank k over M. Moreover, the transition functions $g^F_{\beta\alpha}$ for this vector bundle are found by restricting the transition functions $g^E_{\beta\alpha}$ for E: for $x \in U_\alpha \cap U_\beta$,

$$g^E_{\beta\alpha}(x) = \begin{bmatrix} g^F_{\beta\alpha}(x) & \star \\ 0 & \star \end{bmatrix} \in GL(r,\mathbb{R})$$

where $g^F_{\beta\alpha}(x) \in GL(k,\mathbb{R})$.

Proposition 8.3. *Let $\pi : E \to M$ be a smooth vector bundle of rank r over a smooth manifold M. Let $\{F_p : p \in M\}$ be a collection of k-dimensional linear subspaces F_p of E_p and set $F = \cup_p F_p \subset E$. Then F is a smooth subbundle of E of rank k if and only if for each $p \in M$, there is an open neighborhood U of p in M and smooth sections $s_1, \ldots, s_k$ of $\pi : \pi^{-1}(U) = E|_U \to U$ such that for each $q \in U$, the collection $\{s_i(q)\}_{i=1}^k$ form a basis of F_q.*

Example 8.4. The universal line bundle

$$E = \{(l,v) : l \in P_n(\mathbb{R}), v \in l\} \subset P_n(\mathbb{R}) \times \mathbb{R}^{n+1}$$

is a smooth subbundle of the product bundle. For any $l \in P_n(\mathbb{R})$, $l \in U_i$ for some $i \in \{1, \dots, n+1\}$, where $U_i = \{[x_1, \dots, x_{n+1}] \in P_n(\mathbb{R}) : x_i \neq 0\}$. On U_i, we define $s_i : U_i \to E|_{U_i}$ by

$$s_i([y_1, \dots, y_{i-1}, 1, y_i, \dots, y_n]) = ([y_1, \dots, y_{i-1}, 1, y_i, \dots, y_n], (y_1, \dots, y_{i-1}, 1, y_i, \dots, y_n).$$

Then s_i is a smooth section of $U_i \times \mathbb{R}^{n+1} \to U_i$, and $E_l = \mathbb{R}s_i(l)$ for any $l \in U_i$. By Proposition 8.3, E is a rank 1 smooth subbundle of $P_n(\mathbb{R}) \times \mathbb{R}^{n+1}$.

Definition 8.5 (Distribution)**.** Let M be a smooth manifold. A *smooth distribution of dimension k on M* is a collection $\{F_p \subset T_pM : p \in M\}$ of k-dimensional subspaces F_p of T_pM such that $F = \cup_p F_p$ is a smooth subbundle of rank k of TM.

Remark 8.6. By Proposition 8.3, a collection $\{F_p \subset T_pM : p \in M\}$ of k-dimensional subspaces F_p of T_pM is a smooth distribution if and only if for each $p \in M$, there is an open neighborhood U of p and smooth vector fields $X_1, \dots, X_k$ on U such that for each $q \in U$, the list $\{X_1(q), \dots, X_k(q)\}$ forms a basis for F_q.

Remark 8.7. Let $C^\infty(M, F)$ denote the space of smooth sections of the subbundle $F \to M$. Note that $C^\infty(M, F)$ is a $C^\infty(M)$-submodule of the space $C^\infty(M, TM)$ of smooth sections of TM, that is, the space of smooth vector fields on M.

Definition 8.8. Let F be a smooth distribution of dimension k on a smooth manifold M of dimension n.

(i) We say that F is *involutive* if $C^\infty(M, F)$ is a Lie subalgebra of $(C^\infty(M, TM), [-, -])$.

(ii) We say that F is *completely integrable* if for each p in M, there is a chart (U, ϕ) for M around p such that for each $q \in U$, the subspace F_q is spanned by the list $\{\frac{\partial}{\partial x_1}(q), \dots, \frac{\partial}{\partial x_k}(q)\}$, where $(x_1, \dots, x_n)$ are local coordinates on U.

Remark 8.9. Note that F is completely integrable if and only if for each $p \in M$, there is a k-dimensional submanifold $S \subset M$ such that $p \in S$ and for any $q \in S$, the subspace $T_qS = F_q$.

Example 8.10. We see that a smooth distribution F has the same dimension as M if and only if $F = TM$. And of course F is involutive and completely integrable.

Example 8.11. If the dimension of F is 1, then F is both involutive and completely integrable. For each point $p \in M$, there is an open neighborhood U of p in M and a smooth vector field X on U such that $F_q = \mathbb{R}X(q)$ for each $q \in U$. There is an integral curve of X on this neighborhood showing that F is completely integrable. Moreover, to see that F is involutive, we note that any smooth section of F is locally a multiple of X and hence

$$[fX, gX] = (fX(g) - gX(f))X.$$

Lemma 8.12. *If F is completely integrable then F is involutive.*

Proof. Suppose that X and Y are smooth sections of F. On a coordinate chart (U, ϕ), we may write

$$X = \sum_{i=1}^{k} a_i \frac{\partial}{\partial x_i}$$
$$Y = \sum_{i=1}^{k} b_i \frac{\partial}{\partial x_i}$$

for some smooth functions $a_i, b_i \in C^\infty(U)$. Then we compute that

$$[X,Y] = \sum_{i,j=1}^{k} \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

belongs to the span of $\{\frac{\partial}{\partial x_1}(q), \ldots, \frac{\partial}{\partial x_k}(q)\}$. □

The converse is also true:

Theorem 8.13 (Frobenius)**.** *A smooth distribution F on a smooth manifold is completely integrable if and only if F is involutive.*

Proof. A reference is [Bo, Chapter IV, Section 8]. □

Operations on vector bundles

Let $\pi : E \to M$ be a smooth vector bundle of rank r over a smooth manifold M. We will construct a smooth vector bundle $\pi^* : E^* \to M$ called the dual bundle, whose fibers are given by $E_p^* = (E_p)^*$.

Let $\pi : E \to M$ be a smooth vector bundle of rank r over a smooth manifold M. Let E^* denote the set

$$E^* = \bigcup_{p \in M} E_p^*.$$

Define $\pi^* : E^* \to M$ such that $\pi^*(E_p^*) = \{p\}$. We wish to equip E^* with the structure of a smooth manifold.

1. Suppose that $\{U_\alpha : \alpha \in I\}$ is an open cover of M and $h_\alpha^E : \pi^{-1}(U_\alpha) = E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^r$ are local trivializations of E. Let $\{e_1, \ldots, e_r\}$ be the standard basis of $\mathbb{R}^r$, and define $s_{\alpha i} : U_\alpha \to \pi^{-1}(U_\alpha)$ by $h_\alpha^{-1}(x, e_i)$. Then $\{s_{\alpha 1}, \ldots, s_{\alpha r}\}$ is a C^∞ frame of $E|_{U_\alpha} \to U_\alpha$. Suppose that $U_\alpha \cap U_\beta \neq \emptyset$. Then there exists a C^∞ map $g_{\beta\alpha}^E : U_\alpha \cap U_\beta \to GL(r, \mathbb{R})$ such that

$$s_{\alpha j}(x) = \sum_{i=1}^{r} s_{\beta i}(x) g_{\beta\alpha}^E(x)_{ij}.$$

The transition function $h_\beta^E \circ (h_\alpha^E)^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r$ is given by

$$\begin{aligned} h_\beta^E \circ (h_\alpha^E)^{-1}(x, v) &= (h_\beta^E)(x, \sum_{j=1}^{r} v_j s_{\alpha j}(x)) = h_\beta^E(x, \sum_{i,j=1}^{r} v_j s_{\beta i}(x) g_{\beta\alpha}^E(x)_{ij}) \\ &= h_\beta^E(x, \sum_{i=1}^{r} u_i s_{\beta i}(x)) = (x, u_i) \end{aligned}$$

where $u_i = \sum_{j=1}^{r} g_{\beta\alpha}^E(x)_{ij} v_j$. So the transition function is given by

$$h_\beta^E \circ (h_\alpha^E)^{-1}(x, v) = (x, g_{\beta\alpha}^E(x) v).$$

2. Let $\Gamma(U_\alpha, E^*|_{U_\alpha})$ denote the set of maps $s : U_\alpha \to (\pi^*)^{-1}(U_\alpha) = \cup_{x \in U_\alpha} E_x^*$ such that $s(x) \in E_x^*$. For any $x \in U_\alpha$, let $\{s_{\alpha 1}^*(x), \ldots, s_{\alpha r}^*(x)\}$ be the basis of E_x^* dual to the basis $\{s_{\alpha 1}(x), \ldots, s_{\alpha r}(x)\}$ of E_x:

$$\langle s_{\alpha i}^*(x), s_{\alpha j}(x) \rangle = \delta_{ij}.$$

Then $s_{\alpha 1}^*, \ldots, s_{\beta r}^* \in \Gamma(U_\alpha, E^*|_{U_\alpha})$, and there is a bijection

$$\Phi_\alpha : U_\alpha \times \mathbb{R}^r \to (\pi^*)^{-1}(U_\alpha), \quad (x, v) \mapsto (x, \sum_{i=1}^{r} v_i s_{\alpha i}^*(x)).$$

We equip $(\pi^*)^{-1}(U_\alpha)$ with the topological structure and C^∞ structure such that the bijection Φ_α is a C^∞ diffeomorphism. Define $h_\alpha^{E^*} := \Phi_\alpha^{-1} : (\pi^*)^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$. Then $\pi^*|_{(\pi^*)^{-1}(U_\alpha)} = \mathrm{pr}_1 \circ h_\alpha^{E^*}$ and $h_\alpha^{E^*}|_{E_x^*}$ is a linear isomorphism from E_x to $\{x\} \times \mathbb{R}^r \cong \mathbb{R}^r$ for all $x \in U_\alpha$
3. Suppose that $U_\alpha \cap U_\beta \neq \emptyset$.

$$s_{\beta i}^*(x) = \sum_{i=1}^r \langle s_{\beta i}^*(x), s_{\alpha j}(x)\rangle s_{\alpha j}^*(x) = \sum_{j=1}^r g_{\beta\alpha}^E(x)_{ij} s_{\alpha j}^*(x) = \sum_{j=1}^r s_{\alpha j}^*(x)\left(g_{\beta\alpha}^E(x)^T\right)_{ji}.$$

where A^T denote the transpose of A. Therefore,

$$h_\alpha^{E^*} \circ (h_\beta^{E^*})^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r$$

is given by $h_\alpha^{E^*} \circ (h_\beta^{E^*})^{-1}(x, v) = (x, g_{\beta\alpha}^E(x)^T v)$. Its inverse map

$$h_\beta^{E^*} \circ (h_\alpha^{E^*})^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r$$

is given by

$$h_\beta^{E^*} \circ (h_\alpha^{E^*})^{-1}(x, v) = (x, (g_{\beta\alpha}^E(x)^T)^{-1} v) \tag{8.1}$$

which is a C^∞ diffeomorphism. This shows that the topological structures and C^∞ structures on $(\pi^*)^{-1}(U_\alpha)$ and $(\pi^*)^{-1}(U_\beta)$ defined in Step 2 coincide on their intersection $(\pi^*)^{-1}(U_\alpha \cap U_\beta)$, so we obtain the structure of a C^∞ manifold on E^*. Indeed, by shrinking U_α we may assume that there is a C^∞ atlas on M of the form $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$. Define

$$\tilde{\phi}_\alpha : (\pi^*)^{-1}(U_\alpha) \to \phi_\alpha(U_\alpha) \times \mathbb{R}^r, \quad \tilde{\phi}_\alpha(x, \sum_{i=1}^r v_i s_{\alpha i}^*(x)) = (\phi_\alpha(x), (v_1, \dots, v_r)).$$

Then $\{\left((\pi^*)^{-1}(U_\alpha), \tilde{\phi}_\alpha\right) : \alpha \in I\}$ is a C^∞ atlas for E^*. Moreover, $h_\alpha^{E^*}$ and $h_\beta^{E^*} \circ h_\alpha^{E^*}$ satisfy (i) and (ii) in Definition 4.15, respectively. Finally, (8.1) tells us $g_{\beta\alpha}^{E^*}(x) = (g_{\beta\alpha}^E(x)^T)^{-1}$ for $x \in U_\alpha \cap U_\beta$.

Remark 8.14. The C^∞ structure on E^* is characterized as follows. Let $\Gamma(M, E^*)$ denote the set of maps $\phi : M \to E^* = \cup_{x\in M} E_x^*$ such that $\phi(x) \in E_x^*$. We say $\phi \in \Gamma(M, E^*)$ is a smooth section of $E^* \to M$ if, for every smooth section $s : M \to E$, the function $\langle \phi, s\rangle : M \to \mathbb{R}$ is smooth. Equivalently, given C^∞ frame $\{s_{\alpha 1}, \dots, s_{\alpha r}\}$ of $E|_{U_\alpha}$, we declare that $\{s_{\alpha 1}^*, \dots, s_{\alpha r}^*\}$ is a C^∞ frame of $E^*|_{U_\alpha}$. For any $\phi \in \Gamma(U_\alpha, E^*|_{U_\alpha})$ we may write

$$\phi(x) = \sum_{i=1}^r a_i(x) s_{\alpha i}^*(x), \quad x \in U_\alpha.$$

ϕ is a smooth section, i.e., $\phi \in C^\infty(U_\alpha, E^*|_{U_\alpha})$, if and only if $a_1, \dots, a_r$ are smooth functions on U_α.

Let F be another smooth vector bundle over M. We may apply operations on vector spaces to construct new smooth vector bundles. For example, we can construct $E \oplus F$ and $E \otimes F$ whose fibers are given by $E_p \oplus F_p$ and $E_p \otimes F_p$ respectively. As another example, we can take $\mathrm{Hom}(E, F)$ whose fibers are given by $\mathrm{Hom}(E, F)_p = \mathrm{Hom}(E_p, F_p)$. Note that $\mathrm{Hom}(E, F) \simeq E^* \otimes F$. We can also take the k-th exterior power $\Lambda^k E$, where $k \leq r$.

In each above example, the smooth structure is given by the following. For each point $p \in M$, we take a neighborhood U of p such that there is a C^∞ frame $\{e_1, \ldots, e_r\}$ for $E|_U$ and a C^∞ frame $\{f_1, \ldots, f_s\}$ for $F|_U$.

- The dual frame $\{e_1^*, \ldots, e_r^*\}$ is a C^∞ frame for $E^*|_U$.
- $\{e_1, \ldots, e_r, f_1, \ldots, f_s\}$, we get a C^∞ frame for $(E \oplus F)|_U$.
- $\{e_i \otimes f_j : 1 \le i \le r, 1 \le j \le s\}$ is a C^∞ frame for $(E \otimes F)|_U$.
- $\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \le i_1 < \cdots < i_k \le r\}$ is a C^∞ frame of $\Lambda^k E$. (Here $k \le r$.)

9. Wednesday, October 7, 2015

Definition 9.1. Let M be a smooth manifold. The *cotangent space* at $p \in M$ is the space $T_p^*M := (T_pM)^*$, the dual vector space of the tangent space T_pM to M at p. A *cotangent vector* at $p \in M$ is an element of T_p^*M. The *cotangent bundle* of M is $T^*M := (TM)^*$, the dual of the tangent bundle TM of M.

Definition 9.2. Let M be a smooth manifold.

(i) A *smooth (r, s)-tensor* on M is a smooth section of

$$T_s^r M := (TM)^{\otimes r} \otimes (T^*M)^{\otimes s}.$$

(ii) A *smooth s-form* on M is a smooth section of $\Lambda^s T^*M$.

Example 9.3. A vector field is a (1,0)-tensor. An s-form is a particular type of $(0, s)$-tensor. A 1-form is the same as a $(0, 1)$-tensor.

Example 9.4. Let $f : M \to \mathbb{R}$ be smooth. Then for any point $p \in M$, the differential df_p is a linear map $df_p : T_pM \to \mathbb{R}$. It follows that $df_p \in T_p^*M$. Suppose that (U, ϕ) is a chart for M and $\phi = (x_1, \ldots, x_n)$ are local coordinates. Then

$$\langle df, \frac{\partial}{\partial x_i} \rangle = \frac{\partial f}{\partial x_i}$$

are smooth functions on U. This shows that df is a smooth section of T^*M, i.e., df is a smooth 1-form on M. The 1-form df is called the *differential* of f.

We now study tensors in local coordinates. Let (U, ϕ) be a chart for M such that $\phi = (x_1, \ldots, x_n)$. Then we know that $\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\}$ is a smooth frame for $TM|_U = TU$. The differentials dx_i of the coordinate functions are smooth sections of $T^*M|_U = T^*U$ and

$$dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij}.$$

So $\{dx_1, \ldots, dx_n\}$ is a C^∞ frame of T^*U dual to the C^∞ frame $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ For any smooth function $f : U \to \mathbb{R}$, we may write

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i.$$

More generally, any smooth (r, s)-tensor can be written in terms of the local frames:

$$\sum_{\substack{1 \le i_1, \cdots, i_r \le n \\ 1 \le j_1, \cdots, j_s \le n}} a_{j_1 \cdots j_s}^{i_1 \cdots i_r} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}$$

where $a_{j_1 \cdots j_s}^{i_1 \cdots i_r} \in C^\infty(U)$.

Pullback of $(0, s)$ tensors under a C^∞ map

Definition 9.5. Let $\phi : M \to N$ be a smooth map between smooth manifolds. Let p be a point of M. Then $d\phi_p : T_pM \to T_{\phi(p)}N$ is a linear map. We get a dual linear map $d\phi_p^* : T^*_{\phi(p)}N \to T^*_pM$. Then for any $(0, s)$-tensor T on N, we let ϕ^*T denote the $(0, s)$-tensor of M described by

$$\phi^*T(p) = (d\phi_p^*)^{\otimes s}(T(\phi(p))).$$

Definition 9.6. We let $\Omega^s(N)$ denote the space of smooth s-forms on N, that is, the space of smooth sections of $\Lambda^s T^*N$. The above definition implies that we may pull back s-forms.

Lemma 9.7. *For any smooth function $f : N \to \mathbb{R}$, we have*

$$\phi^*(df) = d(\phi^*f).$$

Proof. For any $p \in M$, we compute

$$(\phi^*df)(p) = d\phi_p^*(df_{\phi(p)}) = df_{\phi(p)} \circ d\phi_p = d(f \circ \phi)_p = d(\phi^*f)(p).$$

□

Example 9.8. Let $\phi : (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ be the map

$$\phi(r, \theta) = (r\cos\theta, r\sin\theta).$$

Note $\phi^*x = r\cos\theta$ and $\phi^*y = r\sin\theta$. Then

$$\phi^*dx = d(\phi^*x) = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta$$

and

$$\phi^*dy = d(\phi^*y) = d(r\sin\theta) = \sin\theta dr + r\cos\theta d\theta.$$

We also compute that

$$\phi^*(dx \wedge dy) = rdr \wedge d\theta.$$

Pullback and pushforward of tensors under a C^∞ diffeomorphism

Definition 9.9. Let $\phi : M \to N$ be a smooth diffeomorphism. It follows that $d\phi_p : T_pM \to T_{\phi(p)}N$ is an invertible linear map with inverse $d(\phi^{-1})_{\phi(p)}$. Then we get a map $\phi^* : C^\infty(N, T^r_sN) \to C^\infty(M, T^r_sM)$ called the *pullback* described by

$$\phi^*T(p) = [(d(\phi^{-1})_{\phi(p)})^{\otimes r} \otimes (d\phi_p^*)^{\otimes s}]T(\phi(p))$$

We also get a map $\phi_* : C^\infty(M, T^r_sM) \to C^\infty(N, T^r_sN)$ called the *pushforward* described by $\phi_* = (\phi^{-1})^*$.

Example 9.10. If X is a smooth vector field, then

$$\phi_*X(q) = d\phi_{\phi^{-1}(q)}X(\phi^{-1}(q))$$

for any $q \in N$.

Lemma 9.11. *If $\phi : M_1 \to M_2$ and $\psi : M_2 \to M_3$ are smooth maps.*

(i) *Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.*
(ii) *If ϕ, ψ are diffeomorphisms, then $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.*

Lie derivatives on tensors

Let X be a smooth vector field on M. We have already defined $L_Xf = X(f)$ for $f : M \to \mathbb{R}$ a smooth function. We have also defined $L_X(Y) = [X, Y]$ for a smooth vector field Y on M. Now we want to define L_XT for any smooth tensor T.

Recall from before that if $\phi_t : U \to M$ is the local flow of X around $p \in M$, then

$$L_X f(p) = \frac{d}{dt}\Big|_{t=0} (\phi_t^* f)(p)$$

and

$$L_X Y(p) = \frac{d}{dt}\Big|_{t=0} (\phi_t^* Y)(p).$$

Note that $\phi_t^* = (\phi_t^{-1})_* = (\phi_{-t})_*$.

Definition 9.12. Let M be a smooth manifold and let X be a smooth vector field. We can define the *Lie derivative* with respect to X to be the map $L_X : C^\infty(M, T_s^r M) \to C^\infty(M, T_s^r M)$ by the rule

$$L_X T(p) := \frac{d}{dt}\Big|_{t=0} (\phi_t^* T)(p)$$

where $\phi_t : U \to M$ is the local flow of X.

Lemma 9.13. *The Lie derivative satisfies the following properties*

(i) *For a smooth function f, we have $L_X f = X(f)$.*
(ii) *For a smooth vector field Y, we have $L_X Y = [X, Y]$.*
(iii) *For a (0,1)-tensor α and Y a vector field, we have*

$$(L_X \alpha)(Y) = L_X(\alpha(Y)) - \alpha(L_X Y) = X(\alpha(Y)) - \alpha([X, Y]).$$

(iv) *For tensors S and T, we have*

$$L_X(S \otimes T) = L_X(S) \otimes T + S \otimes L_X(T).$$

In particular, if f is a smooth function, then

$$L_X(fT) = X(f)T + f L_X T$$

Proof. To see (iii), we can check that

$$\phi_t^*(\alpha(Y)) = (\phi_t^* \alpha)(\phi_t^*(Y)).$$

For (iv), we can check that

$$\phi_t^*(S \otimes T) = \phi_t^* S \otimes \phi_t^* T.$$

□

Remark 9.14. Alternatively, one can use properties (i) through (iv) to define the Lie derivative.

Lemma 9.15. $L_X \circ L_Y - L_Y \circ L_X = L_{[X,Y]}$.
This means that the map $L : C^\infty(M, TM) \to \mathfrak{gl}(C^\infty(M, T_s^r M))$ given by $X \mapsto L_X$ is a Lie algebra homomorphism.

Proof. Assignment 5 (1). □

Exterior derivative on forms

Definition 9.16. Define $d : \Omega^s(M) \to \Omega^{s+1}(M)$ to be the unique $\mathbb{R}$-linear map satisfying

(i) If f is a smooth function on M, then df is the differential of f.
(ii) For any smooth function f on M, we have $ddf = 0$.
(iii) (Leibniz rule): If α is an r-form and β is an s-form, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta.$$

In terms of local coordinates, we have the following. If α is an s-form and (U, ϕ) is a local coordinate chart, then we may write

$$\alpha = \sum_{1 \le j_1 < \cdots j_s \le n} a_{j_1 \cdots j_s} dx_{j_1} \wedge \cdots \wedge dx_{j_s}$$

and we compute

$$d\alpha = \sum_{1 \le j_1 < \cdots j_s \le n} da_{j_1 \cdots j_s} \wedge dx_{i_1} \wedge \cdots \wedge dx_{j_s}$$

Proposition 9.17. *Let ω be an s-form on M. Then we have the following.*

(i) $dd\omega = 0$.

(ii) *If $\phi : M' \to M$ is a smooth map, then $d(\phi^*\omega) = \phi^*(d\omega)$, that is, d commutes with pullbacks.*

(iii) *If X is a smooth vector field on M, then $d(L_X\omega) = L_X(d\omega)$, that is, d commutes with Lie derivatives.*

(iv) *For an s-form ω and vector fields $X_0, \dots, X_s$, we compute*

$$d\omega(X_0, \dots, X_s) = \sum_{i=0}^{s} (-1)^i X_i(\omega(X_0, \dots, \hat{X}_i, \dots, X_s)) + \sum_{0 \le i,j \le s} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_s).$$

Proof. The proofs of (i) and (ii) are straightforward. Taking $\phi = \phi_t$ in (ii), we get (iii). The proof of (iv) is Assignment 5 (3). □

Interior derivatives on forms

Definition 9.18. Let X be a smooth vector field on a smooth manifold M. Define $i_X : \Omega^s(M) \to \Omega^{s-1}(M)$ by the rules

(i) $i_X f = 0$ for a smooth function $f : M \to \mathbb{R}$ and

(ii) For an s-form α, we have $i_X\alpha(X_1, \dots, X_{s-1}) = \alpha(X, X_1, \dots, X_{s-1})$.

Lemma 9.19. *We have the following.*

(i) $i_X \circ i_X = 0$

(ii) $i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge i_X\beta$.

(iii) *(Cartan's formula): We have $d \circ i_X + i_X \circ d = L_X$.*

Proof. (i) and (ii) are straightfoward to check. (iii) is Assignment 5 (2a). □

10. Monday, October 12, 2015

Riemannian Metrics

Definition 10.1. Let M be a smooth manifold. A *Riemannian metric* g on M is a smooth $(0,2)$-tensor such that for any $p \in M$, $g(p) : T_pM \times T_pM \to \mathbb{R}$ is a inner product on T_pM. We say such a pair (M, g) is a *Riemannian manifold.*

The tensor bundle $T_2^0 M$ can be written as a direct sum of two C^∞ subbundles:

$$T_2^0 M = (T^*M)^{\otimes 2} = S^2(T^*M) \oplus \Lambda^2(T^*M)$$

where $S^2(T^*M)$ is the symmetric square of T^*M.

Let $n = \dim(M)$. For any $p \in M$,

- $(T_p^*M)^{\otimes 2}$ is the space of bilinear forms on T_pM, which is n^2 dimensional;

- $S^2T_p^*M$ is the space of symmetric bilinear forms on T_pM, which is $\frac{1}{2}n(n+1)$ dimensional;
- $\Lambda^2T_p^*M$ is the space of skew-symmetric bilinear forms on T_pM, which is $\frac{1}{2}n(n-1)$ dimensional.

Let $\Omega \subset C^\infty(M, S^2T^*M)$ denote the space of Riemannian metrics on M. Then we claim that Ω is a convex subset. This is because if $g_0, g_1 \in \Omega$, then $(1-t)g_0 + tg_1$ is a Riemannian metric for $t \in [0,1]$. In particular, we see that Ω is contractible.

We now discuss Riemannian metrics in local coordinates. Let (U, ϕ) be a chart for M and write $\phi = (x_1, \dots, x_n)$. Then $\{dx_1, \dots, dx_n\}$ is a C^∞ frame for $T^*M|_U = T^*U$. If we let

$$dx_i dx_j = \frac{1}{2}(dx_i \otimes dx_j + dx_j \otimes dx_i)$$

then we see that $\{dx_i dx_j : 1 \leq i \leq j \leq n\}$ is a C^∞ frame for $S^2T^*M|_U$. Then we know that on U, we may write

$$g = \sum_{i,j} g_{ij} dx_i dx_j$$

for some smooth functions g_{ij}, where $g_{ij} = g_{ji}$. For any p, the collection $(g_{ij}(p))$ forms a symmetric, positive definite, $n \times n$ matrix with entries in $\mathbb{R}$.

Example 10.2. Let $M = \mathbb{R}^n$. Then we let $g_0(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \delta_{ij}$. This is called the Euclidean metric. In terms of global coordinates $(x_1, \dots, x_n)$ on $\mathbb{R}^n$,

$$g_0 = dx_1^2 + \cdots + dx_n^2.$$

Example 10.3. On $\mathbb{R}^2$, let (x, y) be the cartesian coordinates, so that the Eulidean metric g_0 can be written as $g_0 = dx^2 + dy^2$. The polar coordinates (r, θ), which are local coordinates around any point in $\mathbb{R}^2 - \{(0,0)\}$, are related to (x, y) by

$$x = r\cos\theta, \quad y = r\sin\theta$$

In terms of the polar coordinates, the Euclidean metric is of the form

$$g_0 = Edr^2 + F(drd\theta + d\theta dr) + Gd\theta^2 = Edr^2 + 2Fdrd\theta + Gd\theta^2,$$

where

$$E = g_0(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}), \quad F = g_0(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}), \quad G = g_0(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}).$$

We have

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} = \frac{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}}{\sqrt{x^2+y^2}}, \\ \frac{\partial}{\partial \theta} &= -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \end{aligned}$$

We compute that $E = 1$, $F = 0$ and $G = r^2$. It follows that

$$g_0 = dr^2 + r^2 d\theta.$$

$\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ is an C^∞ orthonormal frame for $T\mathbb{R}^2$.
$\{\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}\}$ is a C^∞ orthonormal frame for $T\mathbb{R}^2|_{\mathbb{R}^2-\{(0,0)\}}$.

Example 10.4. On $\mathbb{R}^3$, the Euclean metric is $g_0 = dx^2 + dy^2 + dz^2$ in terms of the cartesian coordinates (x, y, z). The spherical coordinates (ρ, ϕ, θ) are local coordinates around any point in $U := (\mathbb{R}^2 - \{(0,0)\}) \times \mathbb{R}$, the complement of the z-axis $x = y = 0$; they are related to the cartesian coordinates by

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

We find that

$$\begin{aligned} \frac{\partial}{\partial \rho} &= \frac{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}}{\sqrt{x^2+y^2+z^2}} \\ \frac{\partial}{\partial \theta} &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \phi} &= \frac{1}{\sqrt{x^2+y^2}}\left(xz\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} - (x^2+y^2)\frac{\partial}{\partial z}\right). \end{aligned}$$

$\rho = \sqrt{x^2+y^2+z^2}$ is a smooth function on U; indeed it is a smooth function on $\mathbb{R}^3 - \{(0,0,0)\}$. Although ϕ and θ are well-defined only locally but not globally on U, the above computations show that $\frac{\partial}{\partial \rho}$, $\frac{\partial}{\partial \theta}$, $\frac{\partial}{\partial \phi}$ are well-defined C^∞ vector fields on U and form a C^∞ frame for $T\mathbb{R}^3|_U$; $d\phi$ and $d\theta$ are well-defined, smooth 1-forms on U, and $\{d\rho, d\theta, d\phi\}$ is a C^∞ frame for $T^*\mathbb{R}^3|_U$.

We compute that

$$g_0 = d\rho^2 + \rho^2 d\phi^2 + \rho^2 \sin^2\phi d\theta^2.$$

$\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ is a C^∞ orthonormal frame for $T\mathbb{R}^3$.
$\{\frac{\partial}{\partial \rho}, \frac{1}{\rho}\frac{\partial}{\partial \phi}, \frac{1}{\rho \sin\phi}\frac{\partial}{\partial \theta}\}$ is a C^∞ orthonormal frame for $T\mathbb{R}^3|_U$.

Let $f : M \to N$ be a smooth map between smooth manifolds. If g is a Riemannian metric on N, then $g \in C^\infty(N, S^2(T^*N))$, so $f^*g \in C^\infty(M, S^2T^*M)$. Given $p \in M$, $(f^*g)(p)$ is an inner product on T_pM iff $df_p : T_pM \to T_{f(p)}M$ is injective iff f is an immersion at p. Therefore, if f is an immersion then f^*g is a Riemannian metric on M.

Definition 10.5. Let $f : M \to N$ be a smooth immersion and let g be a Riemannian metric on N. Then f^*g is a Riemannian metric on M called the *pullback*.

Example 10.6. Let $i_r : S^2(r) \to \mathbb{R}^3$. Then $g_{can} := i_r^* g_0$ is known as the *canonical metric* or *round metric* on the sphere of radius r.

It is convenient to use the coordinates

$$x = r \sin\phi \cos\theta, \quad y = r \sin\phi \sin\theta, \quad z = r \cos\phi.$$

Then we find that

$$g_{can} = i_r^* g_0 = r^2(d\phi^2 + \sin^2\phi d\theta^2)$$

Definition 10.7. Let $f : (M, g_1) \to (N, g_2)$ be a smooth map between Riemannian manifolds. We say that f is

(i) an *isometric immersion* (resp. *embedding*) if f is an immersion (resp. embedding) and $f^*g_2 = g_1$ (in other words, if the differential preserves the inner product).
(ii) a *(local) isometry* if f is a (local) diffeomorphism and $f^*g_2 = g_1$.

Suppose that $i : (M_1, g_1) \hookrightarrow (M_2, g_2)$ is an isometric embedding. Then $i(M_1)$ is a Riemannian submanifold of (M_2, g_2). This means that it is a submanifold when equipped with the Riemannian metric given by pulling back the metric on M_2 under inclusion.

Example 10.8. Let $i_r : S^n(r) \to \mathbb{R}^{n+1}$. Then $g_{can} = i_r^* g_0$ is the round metric on the n-sphere of radius $r > 0$.

Example 10.9. Let $A \in GL(n, \mathbb{R})$. Then A defines an invertible linear map $A : \mathbb{R}^n \to \mathbb{R}^n$. In particular, A is a smooth diffeomorphism. Then we can pull back the Euclidean metric. We find that

$$\begin{aligned} A^* g_0 &= \sum_i d\Big(\sum_j A_{ij} dx_j\Big) d\Big(\sum_k A_{ik} dx_k\Big) = \sum_{j,k} \Big(\sum_i A_{ij} A_{ik}\Big) dx_j dx_k \\ &= \sum_{j,k} (A^T A)_{jk} dx_j dx_k. \end{aligned}$$

We see that A is an isometry if and only if $A^* g_0 = g_0$, which happens if and only if $A^T A = I$, which means that $A \in O(n)$.

We will see later the following.

Theorem 10.10. *A smooth map $\phi : (\mathbb{R}^n, g_0) \to (\mathbb{R}^n, g_0)$ is an isometry if and only if ϕ is a rigid motion, i.e. $\phi(x) = Ax + b$ for some $A \in O(n)$ and $b \in \mathbb{R}^n$.*

Example 10.11. Let $A \in O(n+1)$. Then $A(S^n(r)) = S^n(r)$. It follows that the restriction $A : (S^n(r), g_{can}) \to (S^n(r), g_{can})$ is an isometry. We will see later that, these are all of the isometrics of the round sphere.

Example 10.12. Let $\phi : \mathbb{R} \to S^1$ be the map $\phi(t) = (\cos t, \sin t)$. This is a smooth local diffeomorphism. On $\mathbb{R}$, we have the metric dt^2 and on $\mathbb{R}^2$, we have the metric $dx^2 + dy^2$, which induces the metric g_{can} on S^1. Then we find that

$$\phi^* g_{\text{can}} = (i \circ \phi)^* (dx^2 + dy^2) = (-\sin t dt)^2 + (\cos t dt)^2 = dt^2.$$

It follows that $\phi : (\mathbb{R}, dt^2) \to (S^1, g_{\text{can}})$ is a local isometry.

Definition 10.13. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds and let $M_1 \times M_2$ denote the product manifold. For $i = 1, 2$, let $\pi_i : M_1 \times M_2 \to M_i$. We define the *product metric* on $M_1 \times M_2$ to be

$$g_1 \times g_2 = \pi_1^* g_1 + \pi_2^* g_2.$$

In this way, the metric on $T_{(p_1,p_2)}(M_1 \times M_2)$ ensures the space decomposes as an orthogonal sum $T_{p_1} M_1 \oplus T_{p_2} M_2$. This means that

$$(g_1 \times g_2)(p_1, p_2)((u_1, v_1), (u_2, v_2)) = \langle u_1, u_2 \rangle_{p_1} + \langle v_1, v_2 \rangle_{p_2}.$$

Example 10.14. Let T^n denote the torus $\underbrace{S^1 \times \cdots \times S^1}_{n \text{ copies}}$. The flat metric on T is $g = \underbrace{g_{\text{can}} \times \cdots \times g_{\text{can}}}_{n \text{ times}}$. Let $\phi : \mathbb{R}^n \to T^n$ be the map

$$(t_1, \ldots, t_n) \mapsto ((\cos t_1, \sin t_1), \ldots, (\cos t_n, \sin t_n)).$$

Then ϕ is a local isometry from $(\mathbb{R}^n, g_0)$ to (T^n, g).

Definition 10.15. Let M be a smooth manifold (note that we are assuming that M is Hausdorff with a countable basis). A smooth *partition of unity* on M is a collection of smooth functions $\{f_\gamma \in C^\infty(M) : \gamma \in \Gamma\}$ such that

(i) (nonnegative) We have $f_\gamma \geq 0$ for each γ
(ii) (locally finite) The collection $\{\text{supp} f_\gamma : \gamma \in \Gamma\}$ is *locally finite* in the sense that for each $p \in M$, there is a neighborhood W of p such that only finitely many $\text{supp} f_\gamma$ intersect W.
(iii) For each $p \in M$, we have

$$\sum_{\gamma \in \Gamma} f_\gamma(p) = 1.$$

Note that the left hand side is a finite sum by (ii).

Moreover we say that a partition of unity $\{f_\gamma\}$ is *subordinate to an open cover* $\mathcal{A} = \{A_\alpha : \alpha \in I\}$ if for each $\gamma \in \Gamma$, there is an $\alpha \in I$ such that $\text{supp} f_\gamma \subseteq A_\alpha$.

Theorem 10.16. *Let M be a smooth manifold and let $\mathcal{A} = \{A_\alpha : \alpha \in I\}$ be an open cover of M. Then there is a partition of unity $\{f_\gamma : \gamma \in \Gamma\}$ subordinate to the open cover $\mathcal{A}$.*

Proof. See [Bo, Chapter V Section 4]. □

The proofs of the following two propositions rely on Theorem 10.16 and will be presented on the roundtable on October 16.

Proposition 10.17. *Let M be a smooth manifold. Then there is a Riemannian metric on M.*

Proposition 10.18. *Let M be a compact Hausdorff smooth n-manifold. Then M can be smoothly embedded in $\mathbb{R}^{2n+1}$.*

We have the following classical theorems.

Theorem 10.19 (Weak Whitney Embedding)**.** *Let M be a smooth n-manifold (Hausdorff and countable basis). Then M can be smoothly embedded in $\mathbb{R}^{2n+1}$ as a closed submanifold.*

Theorem 10.20 (Strong Whitney Emdedding)**.** *Let M be a smooth n-manifold (Hausdorff with countable basis). Then M can be smoothly embedded in $\mathbb{R}^{2n}$ as a closed submanifold.*

Theorem 10.21 (Nash Embedding Theorem)**.** *Any Riemannian n-manifold can be isometrically embedded in $\mathbb{R}^{n(n+1)(3n+11)/2}$. Any compact Riemannian n-manifold can be isometrically embedded in $\mathbb{R}^{n(3n+11)/2}$.*

11. Wednesday, October 14, 2015

Volume form

Definition 11.1 (Volume Form)**.** Let M be a smooth n-manifold. A *volume form* on M is a smooth n-form ν on M such that $\nu(p) \neq 0$ for any $p \in M$.

Lemma 11.2. *If M is a smooth n-manifold, the following are equivalent.*

(i) *There is a volume form on M.*
(ii) *$\Lambda^n T^* M$ is trivial.*
(iii) *M is orientable.*

Proof. (i)$\Leftrightarrow$ (ii): Item (i) means that there is a global smooth frame for $\Lambda^n T^*M$. This happens if and only if $\Lambda^n T^*M$ is a trivial vector bundle of rank 1 by a previous lemma.
(i)$\Rightarrow$ (iii): Assume that (i) holds. Call the volume form ν. Let $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ be a smooth atlas for M such that each U_α is connected. We define a smooth atlas $\{(U_\alpha, \phi'_\alpha) : \alpha \in I\}$ as follows: On U_α, we may write $\nu = f_\alpha dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha$ where n is the dimension of M and $\phi_\alpha = (x_1^\alpha, \ldots, x_n^\alpha)$ are local coordinates on U_α. We know that $f_\alpha \neq 0$, and U_α is connected. It follows that either $f_\alpha > 0$ or $f_\alpha < 0$ on U_α.

- If $f_\alpha > 0$, define $(U'_\alpha, \phi'_\alpha) = (U_\alpha, \phi_\alpha)$.
- If $f_\alpha < 0$, then let r be the map $r(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$ and define $(U_\alpha, \phi'_\alpha = r \circ \phi_\alpha)$.

Then we can check that $\{(U_\alpha, \phi'_\alpha) : \alpha \in I\}$ defines an orientation on M.
(iii)$\Rightarrow$ (i): Assume that (iii) holds. Suppose that $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ is an orientation on M, that is, $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ is a smooth atlas on M such that $\det d(\phi_\beta \circ \phi_\alpha^{-1}) > 0$ on $\phi_\alpha(U_\alpha \cap U_\beta)$. Equip M with a Riemannian metric g. On U_α, write $\phi_\alpha = (x_1^\alpha, \ldots, x_n^\alpha)$. Then

$$g = \sum_{i,j=1}^{n} g_{ij}^\alpha dx_i^\alpha dx_j^\alpha$$

where $g_{ij}^\alpha = \langle \frac{\partial}{\partial x_i^\alpha}, \frac{\partial}{\partial x_j^\alpha} \rangle \in C^\infty(U_\alpha)$, and $(g_{ij}^\alpha(p))$ is a positive definite symmetric $n \times n$ matrix for every $p \in U_\alpha$.

Define $\nu_\alpha \in \Omega^n(U_\alpha)$ to be $\nu_\alpha = \sqrt{\det(g_{ij}^\alpha)} dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha$. For each $p \in U_\alpha$, we know that $g_{ij}^\alpha(p)$ is a symmetric positive definite matrix and so $\det(g_{ij}^\alpha) : U_\alpha \to (0, \infty)$. Then ν_α is a smooth nowhere zero section of $(\Lambda^n T^*M)|_{U_\alpha}$. If $U_\alpha \cap U_\beta \neq \varnothing$, then

$$g_{kl}^\beta = \langle \frac{\partial}{\partial x_k^\beta}, \frac{\partial}{\partial x_l^\beta} \rangle = \langle \sum_i \frac{\partial x_i^\alpha}{\partial x_k^\beta} \frac{\partial}{\partial x_i^\alpha}, \sum_j \frac{\partial x_j^\alpha}{\partial x_l^\beta} \frac{\partial}{\partial x_j^\alpha} \rangle = \sum_{i,j} \frac{\partial x_i^\alpha}{\partial x_k^\beta} \frac{\partial x_j^\alpha}{\partial x_l^\beta} g_{ij}^\alpha.$$

Write $A_{ij} = g_{ij}^\alpha$ and $B_{kl} = g_{kl}^\beta$ and $C_{ik} = \frac{\partial x_i^\alpha}{\partial x_k^\beta}$. Then $B = C^t A C$. It follows that $\det B = \det A (\det C)^2 \Rightarrow \det B = \det A \sqrt{\det C}$ (since A, B are symmetric and positive definite, and $\det C > 0$). We also have

$$dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha = \det C dx_1^\beta \wedge \cdots \wedge dx_n^\beta.$$

On $U_\alpha \cap U_\beta$,

$$\begin{aligned} \nu_\alpha &= \sqrt{\det A} dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha = \sqrt{\det A} \det C dx_1^\beta \wedge \cdots \wedge dx_n^\beta \\ &= \det B dx_1^\beta \wedge \cdots \wedge dx_n^\beta = \nu_\beta. \end{aligned}$$

□

Remark 11.3. Let (M, g) be an oriented Riemannian manifold of dimension n. Then there is a unique volume form ν compatible with the orientation and the Riemannian metric, namely, the one we constructed. For any $p \in M$, choose an orthonormal basis $(e_1, \ldots, e_n)$ for T_pM compatible with the orientation in the sense that if $\{(U_\alpha, \phi_\alpha)\}$ is an orientation and $\phi_\alpha = (x_1^\alpha, \ldots, x_n^\alpha)$, then $(dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha)_p(e_1, \ldots, e_n) > 0$. Then we let $\nu(p) = e_1^* \wedge \cdots \wedge e_n^*$, where $(e_1^*, \ldots, e_n^*)$ is the dual basis of T_p^*M. This is well-defined because if $(f_1, \dots, f_n)$ is another orthonormal basis which is compatible with the orientation then

$$f_i = \sum_{j=1}^{n} a_{ij} e_j$$

where $a_{ij} = A \in O(n)$ and $\det(A) > 0$ (which means that $A \in SO(n)$) and so

$$f_1^* \wedge \cdots \wedge f_n^* = e_1^* \wedge \cdots \wedge e_n^*.$$

Example 11.4. For $(\mathbb{R}^n, g_0 = dx_1^2 + \cdots + dx_n^2)$, we let $e_i = \frac{\partial}{\partial x_i}$ and $e_i^* = dx_i$ and so $\nu = dx_1 \wedge \cdots \wedge dx_n$.

Example 11.5. Let $j : (S^n, g_{can}) \hookrightarrow (\mathbb{R}^{n+1}, g_0)$ be the round unit sphere isometrically embedded in $\mathbb{R}^{n+1}$. For any $x = (x_1, \dots, x_{n+1}) \in S^n$, we know that

$$T_x S^n = \{v \in \mathbb{R}^{n+1} : x \cdot v = 0\}.$$

Then we find that,

$$\mathrm{vol}_{S^n, g_{can}} = \pm j^*(i_X(dx_1 \wedge \cdots \wedge dx_{n+1}))$$

where $\pm$ depends on the orientation on S^n, and $X = \sum_{j=1}^{n+1} x_j \frac{\partial}{\partial x_j}$.

Example 11.6. More generally, let (N^{n+1}, g) be an oriented Riemannian manifold. Let $j : M^n \hookrightarrow N^{n+1}$ be a submanifold of codimension 1 equipped with the Riemannian metric j^*g. If M is also oriented, then we have volume forms $\nu_M \in \Omega^n(M)$ and $\nu_N \in \Omega^{n+1}(M)$ which are compatible with the orientations and metrics. Suppose that there is a vector field X on N such that for any $p \in M$, we have $|X(p)| = 1$ and $X(p) \perp T_pM$. By replacing X by $-X$ if necessary, we may further assume that $(X(p), e_1, \dots, e_n)$ is an orthonormal basis for T_pN which is compatible with the orientation on N where $e_1, \dots, e_n$ is an orthonormal basis for T_pM compatible with the orientation on M. Then $j^*(i_X \nu_N) = \nu_M$.

Integration on an oriented manifold

Let (M, g) be a smooth n-manifold equipped with an orientation defined by a C^∞ atlas $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$. Let $\phi_\alpha = (x_1^\alpha, \dots, x_n^\alpha)$. Given a smooth n-form ω and a compact subset R of ω, the integral

$$\int_R \omega$$

is characterized by the following properties.

(1) Suppose that R is contained in U_α for some $\alpha \in I$, and let $(x_1^\alpha, \dots, x_n^\alpha)$ be local coordinates on U_α. On U_α, any smooth n-form can be written as $\omega = f_\alpha dx_1^\alpha \wedge \cdots dx_n^\alpha$ for some $f_\alpha \in C^\infty(U_\alpha)$. We define

$$\int_R \omega = \int_{\phi_\alpha(R)} f_\alpha(x) dx_1^\alpha \cdots dx_n^\alpha.$$

(2) If R_1 and R_2 are disjoint compact subsets of M then

$$\int_{R_1 \cup R_2} \omega = \int_{R_1} \omega + \int_{R_2} \omega.$$

(3) If $\omega_1, \omega_2 \in \Omega^n(M)$ and $c_1, c_2 \in \mathbb{R}$ then

$$\int_R (c_1\omega_1 + c_2\omega_2) = c_1 \int_R \omega_1 + c_2 \int_R \omega_2.$$

Let $\{f_\gamma : \gamma \in \Lambda\}$ be a partition of unity subordinate to the open cover $\{U_\alpha : \alpha \in I\}$. Given any $\omega \in \Omega^n(M)$,

$$\int_R \omega = \int_R \sum_{\gamma\in\Lambda} f_\gamma \omega = \sum_{\gamma\in\Lambda} \int_R f_\gamma\omega = \sum_{\gamma\in\Lambda}\int_{R_\gamma} f_\gamma\omega.$$

where $R_\gamma := R \cap \mathrm{Supp}(f_\gamma)$ is a compact set contained in some U_α, so we define $\int_{R_\gamma} f_\gamma\omega$ by (1).

Definition 11.7. Let (M, g) be an oriented Riemannian manifold and let ν_g be a volume form compatible with the orientation and Riemannian metric g. Given a compact set R in M, we define the *volume* of R

$$\mathrm{volume}_g(R) = \int_R \nu_g.$$

Example 11.8. Equip S^2 with the metric $g_{\mathrm{can}} = d\phi^2 + \sin^2\phi d\theta^2$. Let $U = S^2 \setminus \{(0,0,1),(0,0,-1)\}$. An orthonormal frame for $TS^2|_U$ is

$$\frac{\partial}{\partial\phi}, \frac{1}{\sin\phi}\frac{\partial}{\partial\theta}$$

and the dual coframe is $d\phi, \sin\phi d\theta$. (In general, if $e_1,\dots,e_n$ is an orthonormal basis of T_pM which is compatible with the orientation, then $g(p) = e_1^*\otimes e_1^* + \cdots + e_n^*\otimes e_n^*$ and $\nu(p) = e_1^*\wedge\cdots\wedge e_n^*$.) Then we see that the volume form is

$$\nu_{g_{\mathrm{can}}} = \sin\phi d\phi\wedge d\theta,$$

and so

$$\mathrm{volume}_{g_{\mathrm{can}}}(S^2) = \int_0^{2\pi}\int_0^{\pi} \sin\phi d\phi d\theta = 4\pi.$$

Length

Definition 11.9. Let $\gamma : (a,b) \to (M,g)$ be a smooth curve. Then the *length of* γ is

$$\mathrm{length}(\gamma) = \int_a^b \|\gamma'(t)\| dt, \text{ where } \|\gamma'(t)\| = \sqrt{\langle\gamma'(t),\gamma'(t)\rangle_{\gamma(t)}}.$$

Example 11.10. We consider upper half plane $\mathbb{H}^2 = \{(x,y) : \mathbb{R}^2 : y > 0\}$. We endow this with the metric

$$g = \frac{dx^2+dy^2}{y^2}.$$

Pick points $x_1 > x_0$ and $y_1 > y_0 > 0$ in $\mathbb{R}$. Let γ_1 be the straight line from (x_0,y_0) to (x_1,y_0) and let γ_2 be the straight line from (x_0,y_0) to (x_0,y_1):

$$\gamma_1(t) = (t, y_0), \quad t\in(x_0,x_1); \quad \gamma_2(t) = (x_0,t), \quad t \in (y_0,y_1).$$

We compute that $\gamma_1' = \frac{\partial}{\partial x}$,

$$\langle\gamma_1'(t),\gamma_1'(t)\rangle_{\gamma(t)} = \frac{1}{y_0^2}.$$

Hence we find that

$$\mathrm{length}(\gamma_1) = \int_{x_0}^{x_1} |\gamma_1'| dt = \frac{x_1-x_0}{y_0}.$$

On the other hand, we compute that $\gamma_2' = \frac{\partial}{\partial y}$,

$$\langle \gamma_2'(t), \gamma_2'(t) \rangle_{\gamma(t)} = \frac{1}{t^2}$$

and hence

$$\text{length}(\gamma_2) = \int_{y_0}^{y_1} \frac{dt}{t} = \log(y_1/y_0).$$

For any $a > 0$,we can consider $F_a : \mathbb{H} \to \mathbb{H}$ given by $F_a(x, y) = (ax, ay)$ and then

$$F^*g = \frac{d(ax)^2 + d(ay)^2}{(ay)^2} = g.$$

It follows that F_a is an isometry.

12. Monday, October 19, 2015

Distance

Definition 12.1. If (M, g) is a connected Riemannian manifold and $p, q \in M$, then for any p, q in M there exists a piecewise smooth curve $\gamma : [0, 1] \to M$ such that $\gamma(0) = 1$ and $\gamma(1) = q$. We define the *distance from p to q* to be

$$d_g(p, q) = \inf\{\text{length}(\gamma) : \gamma : [0, 1] \to M \text{piecewise smooth}, \gamma(0) = p, \gamma(1) = q\}.$$

From the above definition, it is clear that for $p, q, r \in M$,

- $d_g(p, q) \in [0, \infty)$ and $\text{dist}_g(p, p) = 0$;
- $d_g(p, q) = \text{dist}_g(q, p)$;
- $d_g(p, q) + \text{dist}_g(q, r) \geq \text{dist}_g(p, r)$.

We will see later that if M is Hausdorff then $d_g(p, q) = 0 \Rightarrow p = q$, so that (M, d_g) is a metric space (in the sense of topology). The topology defined by d_g agrees with the topology on M.

Lemma 12.2. *The distance is preserved by isometry. That is, if $\phi : (M_1, g_1) \to (M_2, g_2)$ is an isometry, then*

$$d_{g_1}(p, q) = d_{g_2}(\phi(p), \phi(q)).$$

Proof. Note that $\gamma : I \to M_1$ is a piecewise smooth curve in M_1 if and only if $\phi \circ \gamma : I \to M_2$ is a piecewise smooth curve in M_2, and in this case, we have $\text{length}(\phi \circ \gamma) = \text{length}(\gamma)$. □

Example 12.3. For $(\mathbb{R}^n, g_0)$, $d_{g_0}(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}|$. To see this, by Lemma 12.2 and the fact that rigid motions are isometries, we may assume $\vec{x} = (0, \ldots, 0)$ and $\vec{y} = (d, 0, \ldots, 0)$, where $d \geq 0$. Details are left as an exercise.

Discrete group actions

Definition 12.4. Let G be a group and M a set. We say that G *acts on M on the left* (resp. *on the right*) if there is a map $\phi : G \times M \longrightarrow M$, $\phi(m, g) = \phi_g(m) = g \cdot m$ (resp. $m \cdot g$), satsifying the following (i) and (ii) (resp. (ii)').

(i) If $e \in G$ is the identity of G then $\phi_e : M \to M$ is the identity map.
(ii) (left action) For any $g_1, g_2 \in G$, we have $\phi_{g_1 g_2} = \phi_{g_1} \circ \phi_{g_2}$, i.e. $(g_1 g_2) \cdot m = g_1 \cdot (g_2 \cdot m)$ for all $m \in M$.
(ii)' (right action) For any $g_1, g_2 \in G$, we have $\phi_{g_1 g_2} = \phi_{g_2} \circ \phi_{g_1}$, i.e., $m \cdot (g_1 g_2) = (m \cdot g_1) \cdot g_2$ for all $m \in M$.

Remark 12.5. A left (resp. right) G-action on a set M is the same thing as a group homomorphism $G \to (\mathrm{Perm}(M), \circ)$ given by $g \mapsto \phi_g$ (resp. $g \mapsto \phi_{g^{-1}}$.)

Definition 12.6. Let G be a group and M a topological space. Then we say that G *acts on* M *on the left* (resp. *on the right*) if there is a map $\phi : G \times M \to M$ satisfying (i) and (ii) (resp. (ii)') above and also

(iii) The map $\phi_g : M \to M$ is continuous for each $g \in G$.

Remark 12.7. A left (resp. right) G-action on a topological space M is the same thing as a group homomorphism $G \to (\mathrm{Homeo}(M), \circ)$ given by $g \mapsto \phi_g$ (resp. $g \mapsto \phi_{g^{-1}}$.)

Definition 12.8. Let G be a group and M a topological space and suppose G acts on M on the left. The action of G on M is called *properly discontinuous* if for each point $p \in M$, there is a neighborhood U of p in M such that for each $g \in G \setminus \{e\}$, we have $\phi_g(U) \cap U = \varnothing$.

Remark 12.9. Let U be as in Definition 12.8. If $g_1, g_2 \in G$ are distinct then $\phi_{g_1}(U) \cap \phi_{g_2}(U) = \varnothing$. In particular, a properly discontinuous action is *free* in the sense that if $g \in G$ and $p \in M$, then $g \cdot p = p$ implies that $g = e$.

Proposition 12.10. *If a group G acts on a topological space M properly discontinuously, then the map $\pi : M \to M/G$ is a covering map, where M/G is equipped with the quotient topology.*

Proof. For a point $\bar{p} \in M/G$, there is a $p \in M$ such that $\pi(p) = \bar{p}$. There is an open neighborhood U of p in M such that if g is not the identity, then $g(U) \cap U = \varnothing$. Let $\bar{U}$ be $\pi(U)$. Then $\bar{p} \in \bar{U}$ and $\pi^{-1}(\bar{U})$ is the disjoint union $\sqcup_{g\in G}\phi_g(U)$, where each $\phi_g(U)$ is open. It follows that $\bar{U}$ is an open neighborhood of $\bar{p}$ in M/G. Moreover, the restriction $\pi|_{\phi_g(U)} : \phi_g(U) \to \bar{U}$ is a homeomorphism for any $g \in G$. $\square$

Definition 12.11. Let G be a group and let M be a smooth manifold. We say that G *acts on* M *on the left* (resp. *on the right*) if there is a map $\phi : G \times M \to M$ satisfying (i) and (ii) (resp. (ii)'), and also

(iii)' The map $\phi_g : M \to M$ is a smooth for each $g \in G$.

Remark 12.12. A left (resp. right) G-action on a smooth manifold M is the same thing as a homomorphism $G \to \mathrm{Diffeo}(M)$ given by $g \mapsto \phi_g$ (resp. $g \mapsto \phi_{g^{-1}}$).

Proposition 12.13. *Suppose that a group G acts on a smooth manifold M properly discontinuously. Then*

(i) *There is a unique smooth structure on M/G such that $\pi : M \to M/G$ is a local smooth diffeomorphism.*
(ii) *If h is a Riemannian metric on M and ϕ_g is an isometry of (M, h) for each $g \in G$ (in this case we say that G* acts isometrically on *(M, h)), then there is a unique Riemannian metric $\hat{h}$ on M/G such that $\pi^*\hat{h} = h$.*

Example 12.14. Let $G = \{\pm 1\}$ and let $M = S^n$. Let $\phi_1 = \mathrm{id}$ and let ϕ_{-1} be the antipodal map $A : S^n \to S^n$, $A(x) = -x$. Then G acts properly discontinuously and isometrically on (S^n, g_{can}). It follows that there is a metric $\hat{g}$ on $P_n(\mathbb{R})$ such that $\pi^*\hat{g} = g_{can}$. When $n = 1$, $(P_1(\mathbb{R}), \hat{g})$ is isometric to $S^1(\frac{1}{2})$ (circle of radius $\frac{1}{2}$).

Example 12.15. Let $G = \mathbb{Z}^n$ acts $\mathbb{R}^n$ by

$$(m_1, \dots, m_n) \cdot (x_1, \dots, x_n) \mapsto (x_1 + m_1, \dots, x_n + m_n),$$

where $(m_1, \dots, m_n) \in \mathbb{Z}^n$ and $(x_1, \dots, x_n) \in \mathbb{R}^n$, i.e., $\phi_{(m_1,\dots,m_n)} : \mathbb{R}^n \to \mathbb{R}^n$ is translation by the vector $(m_1, \dots, m_n)$. This action is properly discontinuous and preserves the Euclidean metric g_0, so it descendents to a Riemannian metric $\hat{g}_0$, known as the flat metric, on the quotient $T^n = \mathbb{R}^n/\mathbb{Z}^n$. There is an isometry $(\mathbb{R}^n/\mathbb{Z}^n, \hat{g}_0) \to (S^1(\frac{1}{2\pi}))^n$.

We now discuss orientation.

Definition 12.16. Let V be a real vector space of dimension n. An *orientation* on V is an equivalence class of ordered bases, where two bases are equivalent if the change of coordinates matrix has positive determinant.

Let (U_α, ϕ_α) be a smooth atlas on a smooth manifold M and say it defines an orientation, meaning that the transition functions have positive Jacobian. Choose local coordinates $\phi_\alpha = (x_1^\alpha, \dots, x_n^\alpha)$ around $p \in M$. Then the basis $\{\frac{\partial}{\partial x_i}(q)\}$ defines an orientation on T_qM for each $q \in U_\alpha$.

Definition 12.17. Suppose that $f : M_1 \to M_2$ is a local diffeomorphism between oriented smooth manifolds. We say that f is *orientation preserving* (resp. *orientation reversing*) at $p \in M_1$ if given an ordered basis $(e_1, \dots, e_n)$ of T_pM_1 compatible with the orientation on M_1, the ordered basis $(df_p(e_1), \dots, df_p(e_n))$ of $T_{f(p)}M_2$ is compatible (resp. not compatible) with the orientation on M_2.

We say f is *orientation preserving* (resp. *orientation reversing*) if it is orientation preserving (resp. orientation reversing) at all $p \in M_1$.

Remark 12.18. If M_1 and M_2 are connected, then f is orientation preserving (reversing) at some point $p \in M_1$ if and only if f is orientation preserving (reversing) at all $p \in M_1$.

Example 12.19. The antipodal map $A : S^n \to S^n$ is orientation preserving if and only if n is odd. (cf. Problem (4) of Assignment 6)

Example 12.20. The action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ by translation is orientation preserving.

13. Wednesday, October 21, 2015

Lie groups

Definition 13.1. A *Lie group* G is a group together with the structure of a smooth manifold such that $\lambda : G \times G \to G$ given by $\lambda(x, y) = xy^{-1}$ is a smooth map.

Remark 13.2. From the definition:

- (inverse) The map $G \to G$ given by $x \mapsto x^{-1}$ is smooth.
- (multiplication) The map $G \times G \to G$ given by $(x, y) \mapsto xy$ is smooth.
- (left multiplication) For any $x \in G$, the map $L_x : G \to G$ given by $L_x(y) = xy$ (left multiplication by x) is a smooth map.
- (right multiplication) For any $x \in G$, the map $R_x : G \to G$ given by $R_x(y) = yx$ (right multiplication by x) is a smooth map.

Indeed, G acts on G on the right (resp. on the left) by right (resp. left) multiplication, so L_x and R_x are smooth diffeomorphisms for any $x \in G$.

Example 13.3. $(\mathbb{R}^n, +)$ is a Lie group.

Example 13.4. The set $GL(n,\mathbb{R})$ of invertible $n\times n$ matrices is a smooth manifold with a smooth group operation given by matrix multiplication. This manifold has two connected components, namely, $GL(n,\mathbb{R})_+ = \{A \in GL(n,\mathbb{R}) : \det A > 0\}$ and $GL(n,\mathbb{R})_- = \{A \in GL(n,\mathbb{R}) : \det A < 0\}$. $GL(n,\mathbb{R})_+$ is a connected Lie group. The special linear group $SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : \det A = 1\}$ is a Lie subgroup of $GL(n,\mathbb{R})$.

Example 13.5. The orthogonal group $O(n) = \{A \in GL(n,\mathbb{R}) : A^T A = I_n\}$ is a Lie subgroup of $GL(n,\mathbb{R})$. It has two connected components; $SO(n) = O(n)\cap SL(n,\mathbb{R})$, the connected component of the identity, is a Lie subgroup of $SL(n,\mathbb{R})$.

Definition 13.6. Let G be a Lie group. A tensor T on G is *left* (resp. *right*) *invariant* if $L_x^* T = T$ (resp. $R_x^* T = T$) for each $x \in G$. If a tensor T on G is both left-invariant and right-invariant, then T is called *bi-invariant*.

Remark 13.7. Note that if T is left (resp. right) invariant then T is determined by $T(e)$, the value of T at the identity $e \in G$. In particular:

- A function on G is left (resp. right) invariant if and only if it a constant function.
- A vector field X on G is left (resp. right) invariant if and only if for each $x \in G$, we have $X(x) = d(L_x)_e(X(e))$ (resp. $X(x) = d(R_x)_e(X(e))$).

Let $\mathfrak{X}(G)^L$ (resp. $\mathfrak{X}(G)^R$) denote the space of left (resp. right) invariant vector fields. We have an $\mathbb{R}$-linear isomorphisms $T_eG \xrightarrow{\simeq} \mathfrak{X}(G)^L$ (resp. $T_eG \xrightarrow{\simeq} \mathfrak{X}(G)^R$) described by $\xi \mapsto X_\xi^L$ (resp. $\xi \mapsto X_\xi^R$), where X_ξ^L (resp. X_ξ^R) is the unique left (resp. right) invariant vector field on G such that $X_\xi^L(e) = \xi$ (resp. $X_\xi^R(e) = \xi$). More explicitly, $X_\xi^L(x) = d(L_x)_e(\xi)$ and $X_\xi^R(x) = d(R_x)_e(\xi)$, $x \in G$.

Definition 13.8. Let $F : M \to N$ be smooth and let X be a smooth vector field on M and Y a smooth vector field on N. We say that X and Y are *F-related* if for each $p \in M$, we have $dF_p(X(p)) = Y(F(p))$.

Remark 13.9. If F is a diffeomorphism then X and Y are F-related if and only if $Y = F_*X$.

Remark 13.10. More generally, X and Y are F-related if and only if for each $f \in C^\infty(N)$, we have $X(F^*f) = F^*(Y(f))$.

Proposition 13.11. *Let $F : M \to N$ be smooth, let X_1, X_2 be smooth vector fields on M and let Y_1, Y_2 be smooth vector fields on N. Suppose that X_i and Y_i are F-related. Then $[X_1, X_2]$ and $[Y_1, Y_2]$ are F-related.*

Proof. Let f be a smooth function on N. Then

$$\begin{aligned} [X_1, X_2](F^*f) &= X_1(X_2F^*f) - X_2(X_1F^*f) \\ &= X_1(F^*(Y_2f)) - X_2(F^*(Y_1f)) \\ &= F^*(Y_1Y_2f) - F^*(Y_2Y_1f) \\ &= F^*([Y_1, Y_2]f), \end{aligned}$$

where the second and the third equalities follow from Remark 13.10. By Remark 13.10, $[X_1, X_2]$ and $[Y_1, Y_2]$ are F-related. □

Corollary 13.12. *If F is a smooth diffeomorphism and X_1, X_2 are vector fields on M, then*

$$[F_*X_1, F_*X_2] = F_*[X_1, X_2].$$

Corollary 13.13. *The set of left invariant vector fields $\mathfrak{X}(G)^L$ is a Lie subalgebra of $\mathfrak{X}(G)$. So is the set $\mathfrak{X}(G)^R$.*

Definition 13.14. We define $[-,-] : T_eG \times T_eG \to T_eG$ by

$$(\xi, \eta) \mapsto [X_\xi^L, X_\eta^L](e).$$

We define the *Lie algebra* $\mathfrak{g}$ of G to be T_eG with the above Lie bracket. Then we note that we have an isomorphism $\mathfrak{g} \simeq \mathfrak{X}(G)^L$ as Lie algebras.

Remark 13.15 (Assignment 7 (1))**.** If we let $i : G \to G$ denote the map $g \mapsto g^{-1}$, then $i^2 = \mathrm{id}$ and $di_e(\xi) = -\xi$. We have

$$\begin{array}{ccc} G & \xrightarrow{\ i\ } & G \\ {\scriptstyle L_a}\downarrow & & \downarrow{\scriptstyle R_{a^{-1}}} \\ G & \xrightarrow{\ i\ } & G. \end{array}$$

It follows that $X_\xi^R = -i_* X_\xi^L$. Hence,

$$[X_\xi^R, X_\eta^R] = [i_* X_\xi^L, i_* X_\eta^L] = i_*[X_\xi^L, X_\eta^L] = i_* X_{[\xi,\eta]}^L = -X_{[\xi,\eta]}^R.$$

Proposition 13.16. *The tangent bundle of a Lie group is trivial.*

Proof. Let $\xi_1, \dots, \xi_n$ be a basis of $\mathfrak{g} = T_eG$. Then $X_{\xi_1}^L, \dots, X_{\xi_n}^L$ forms a global C^∞ frame of TG. Let $\phi : G \times \mathfrak{g} \to TG$ be the map

$$(x, \xi) \mapsto (x, X_\xi^L(x)).$$

Then $\phi^{-1} : TG \to G \times \mathfrak{g}$ is a global trivialization of TG. □

Example 13.17. Let $G = (\mathbb{R}^n, +)$. For any $a_1, \dots, a_n \in \mathbb{R}$, the vector field $\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ is bi-invariant. We have

$$\mathfrak{X}(G)^L = \mathfrak{X}(G)^R = \{\sum_{i=1}^n a_i \frac{\partial}{\partial x_i} : (a_1, \dots, a_n) \in \mathbb{R}^n\} \cong \mathbb{R}^n.$$

The Lie bracket on $T_0\mathbb{R}^n$ is trivial. The map ϕ in the proof of Proposition 13.16 is given by

$$\phi : \mathbb{R}^n \times \mathbb{R}^n \to T\mathbb{R}^n, \quad (x, y) \mapsto (x, \sum_{i=1}^n y_i \frac{\partial}{\partial x_i})$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$.

Example 13.18. Let $G = GL(n, \mathbb{R})$. Recall that $\mathfrak{g} = M_n(\mathbb{R})$. For $\xi \in M_n(\mathbb{R})$, then $d(L_A)_{I_n}(\xi) = A\xi$ and $d(R_A)_{I_n}(\xi) = \xi A$. Because of this, we see that

$$\begin{aligned} X_\xi^L(A) &= A\xi = \sum_{i,j} (\sum_k a_{ik}\xi_{kj}) \frac{\partial}{\partial a_{ij}} \\ X_\xi^R(A) &= \xi A = \sum_{i,j} (\sum_k \xi_{ik} a_{kj}) \frac{\partial}{\partial a_{ij}} \end{aligned}$$

The map $\phi : GL(n, \mathbb{R}) \times \mathfrak{g} \to TG = GL(n, \mathbb{R}) \times M_n(\mathbb{R})$ is described by

$$(A, \xi) \mapsto (A, A\xi).$$

Moreover, if H is a Lie subgroup of $G = GL(n, \mathbb{R})$, ϕ restricts to $H \times \mathfrak{h} \subset G \times \mathfrak{h} \to TH \subset TG$. For example, $H = SL(n, \mathbb{R})$, $\mathfrak{h} = \mathfrak{sl}(n, \mathbb{R})$; $H = O(n)$ or $SO(n)$, $\mathfrak{h} = \mathfrak{so}(n)$.

Remark 13.19. This argument of trivializing a bundle will work also for the cotangent bundle of G and more generally for any tensor bundle $T^r_s G$ of G. Indeed, if $E \to M$ is a trivial vector bundle then the dual bundle $E^* \to M$ is also trivial and more generally $E^{\otimes r} \otimes (E^*)^{\otimes s}$ is a trivial vector bundle for any $r, s \in \mathbb{Z}_{\geq 0}$.

Lemma 13.20. *Let ϕ^L_ξ be the flow of X^L_ξ and ϕ^R_ξ the flow of X^R_ξ. Then*

(i) *For each $a \in G$, we have*
$$L_a \circ \phi^L_\xi(t, x) = \phi^L_\xi(t, ax)$$
(ii) *For each $a \in G$, we have*
$$R_a \circ \phi^R_\xi(t, x) = \phi^R_\xi(t, xa).$$

Remark 13.21. This is saying that left (resp. right) multiplication by a carries an integral curve of a left (resp. right) invariant vector field to another integral curve of this vector field.

Proof of Lemma 13.20. It suffices to show that

(a) $(L_a \circ \phi^L_\xi)(0, x) = ax$
(b) $\frac{d}{dt}(L_a \circ \phi^L_\xi)(t, x) = X^L_\xi((L_a \circ \phi^L_\xi)(t, x))$.

To see (a), we note that
$$L_a \circ \phi^L_\xi(0, x) = a \cdot \phi^L_\xi(0, x) = ax.$$
For (b), we note that
$$\begin{aligned}\frac{d}{dt}(L_a \circ \phi^L_\xi)(t, x) &= d(L_a)_{\phi^L_\xi(t,x)}\left(\frac{d}{dt}\phi^L_\xi(t, x)\right) \\ &= d(L_a)_{\phi^L_\xi(t,x)}(X^L_\xi(\phi^L_\xi(t, x))) \\ &= X^L_\xi(L_a \circ \phi^L_\xi(t, x)).\end{aligned}$$
□

Proposition 13.22. *If G is a Lie group and $\xi \in \mathfrak{g}$, then ϕ^L_ξ, ϕ^R_ξ are defined on $\mathbb{R} \times G$.*

Proof. There is an $\epsilon > 0$ and an open neighborhood V of e in G such that $\phi(t, x)$ is defined for $(t, x) \in (-\epsilon, \epsilon) \times V$. By the previous result, we see that $\phi_t(x)$ is defined for $(t, x) \in (-\epsilon, \epsilon) \times G$. Then we see that $\phi_{nt}(x) = \phi_t \circ \cdots \circ \phi_t(x)$ is defined for all $n \in \mathbb{N}$, $t \in (-\epsilon, \epsilon)$, $x \in G$, and hence $\phi(t, x)$ is defined for all $(t, x) \in \mathbb{R} \times G$. □

Example 13.23. If $G = GL(n, \mathbb{R})$ or any Lie subgroup of $GL(n, \mathbb{R})$, then, we see that $X^L_\xi(A) = A\xi$ and also that
$$\begin{aligned}X^L_\xi(A) &= A\xi, \quad \phi^L_\xi(t, A) = A\exp(t\xi), \\ X^R_\xi(A) &= \xi A, \quad \phi^R_\xi(t, A) = \exp(t\xi)A.\end{aligned}$$
Here $\exp(B) = \sum_{n=0}^\infty \frac{B^n}{n!}$, $B \in M_n(\mathbb{R})$. We want to use this observation to extend the definition of the exponential to any Lie group.

Definition 13.24 (Exponential map)**.** If G is a Lie group. Define the *exponential map* $\exp : \mathfrak{g} \to G$ by the rule
$$\xi \mapsto \phi^L_\xi(1, e)$$
where e is the identity of G.

Remark 13.25. We note that $\phi^L_\xi(t,x) = \phi^L_{t\xi}(1,x) = \phi^L_{t\xi}(1, x\cdot e) = x\phi^L_{t\xi}(1,e) = x\exp(t\xi)$. It follows that

$$\phi^L_\xi(t,x) = x\exp(t\xi).$$

In other words

$$(\phi^L_\xi)_t = R_{\exp(t\xi)} : G \to G.$$

14. Wednesday, October 28, 2015

As a special case of Definition 13.6:

Definition 14.1. Let G be a Lie group and let g be a Riemannian metric on G. We say g is *left-invariant* if $L_x^*g = g$ for all $x \in G$. Equivalently, g is left-invariant if and only if for each $x \in G$, $L_x : (G,g) \to (G,g)$ is an isometry.

Remark 14.2. We have a one-to-one correspondence:

$$\{\text{left-invariant metrics on } G\} \leftrightarrow \{\text{inner products on } T_eG\}.$$

Indeed, g is left-invariant if and only if for each $x \in G$ and for each $U, V \in T_xG$,

$$g(x)(U,V) = g(e)(d(L_{x^{-1}})_xU, d(L_{x^{-1}})_xV).$$

Example 14.3. $G = (\mathbb{R}^n, +)$, $g_0 = dx_1^2 + \cdots dx_n^2$. For any $x \in \mathbb{R}^n$, $L_x^*g = R_x^*g = g$. So g is bi-invariant.

Example 14.4. Let

$$G = \{g : \mathbb{R} \to \mathbb{R}, t \mapsto yt + x : x \in \mathbb{R}, y \in (0,\infty)\},$$

that is, the group of proper affine transformations of $\mathbb{R}$. Define multiplication by composition: $g_1(t) = y_1t + x_1$ and $g_2(t) = y_2t + x_2$, then

$$(g_1 \circ g_2)(t) = g_1(y_2t + x_2) = y_1(y_2t + x_2) + x_1 = y_1y_2t + (y_1x_2 + x_1).$$

We may identify G with the upper half plane: $G = \{(x,y) \in \mathbb{R}^2 : y > 0\}$. With this identification, the multiplication is given by

$$(x_1, y_1) \cdot (x_2, y_2) = (y_1x_2 + x_1, y_1y_2).$$

So the multiplication defines a smooth map $G \times G \to G$. The identity element is $e = (0,1)$. The inverse map is given by

$$(x_1, y_1)^{-1} = (-x_1y_1^{-1}, y_1^{-1}),$$

which is smooth. So G is indeed a Lie group.

We note that

$$L_{(a,b)}(x,y) = b(x,y) + (a,0).$$

And hence

$$d(L_{(a,b)})_{(x,y)}(v) = bv.$$

Let g be the unique left-invariant metric on G such that $g(0,1) = dx^2 + dy^2$. We know that g is of the form $g = Edx^2 + 2Fdxdy + Gdy^2$ for some smooth functions E, F, G, where $E(0,1) = G(0,1) = 1$, and $F(0,1) = 0$. We compute

$$L^*_{(a,b)}dx = d(bx + a) = bdx, \quad L^*_{(a,b)}dy = d(by) = bdy.$$

So

$$L^*_{(a,b)}g(x,y) = E(bx + a, by)b^2dx^2 + 2F(bx + a, by)b^2dxdy + G(bx + a, by)b^2dy^2.$$

$$L^*_{(a,b)}g(0,1) = E(a,b)b^2dx^2 + 2F(a,b)b^2dxdy + G(a,b)b^2dy^2.$$

Since g is left-invariant, $(L^*_{(a,b)}g)(0,1) = g(0,1) = dx^2 + dy^2$, so

$$E(a,b) = \frac{1}{b^2}, \quad F(a,b) = 0, \quad G(a,b) = \frac{1}{b^2}.$$

We conclude that

$$g = \frac{dx^2 + dy^2}{y^2}.$$

We find that

$$g = \frac{dx^2 + dy^2}{y^2}.$$

We remark that there is a natural inclusion $G \hookrightarrow \mathrm{Isom}(G, g)$ given by $x \mapsto L_x$.

We can check that this metric is not right-invariant. Indeed

$$R_{(a,b)}(x, y) = (ay + x, by).$$

So we find that

$$R^*_{(a,b)}dx = dR^*_{(a,b)}x = dx + ady$$
$$R^*_{(a,b)}dy = dR^*_{(a,b)}y = bdy.$$

And hence

$$R^*_{(a,b)}g = \frac{(dx + ady)^2 + (bdy)^2}{(by)^2} = \frac{dx^2 + 2adxdy + (a^2 + b^2)dy^2}{b^2y^2}.$$

John Milnor proved the following:

Theorem 14.5 ([Mi, Lemma 7.5])**.** *A connected Lie group admits a bi-invariant Riemannian metric if and only if it is isomorphic to the direct product of a compact Lie group and an additive vector group.*

Definition 14.6 (Adjoint representation)**.** Let G be a Lie group. Given an element $a \in G$, the map $R_{a^{-1}} \circ L_a : G \to G$ is a diffeomorphism sending e to e, and hence we get a linear isomorphism

$$\mathrm{Ad}(a) := d(R_{a^{-1}} \circ L_a)_e : T_eG \to T_eG.$$

This means that we get a group homomorphism

$$\mathrm{Ad} : G \to GL(\mathfrak{g})$$
$$a \mapsto \mathrm{Ad}(a)$$

where $GL(\mathfrak{g})$ is the space of $\mathbb{R}$-linear isomorphisms of $\mathfrak{g}$. This is a representation of G called the *adjoint representation.*

Example 14.7.

(1) Let $G = (\mathbb{R}^n, +)$. For any $a \in \mathbb{R}^n$, $R_{a^{-1}} \circ L_a = \mathrm{id}$ is the identity map, and hence

$$\mathrm{Ad}(a) = \mathrm{id}_{\mathfrak{g}}$$

for each $a \in G$.

(2) More generally, for any abelian Lie group, the adjoint representation is trivial.

(3) Let $G = GL(n, \mathbb{R})$ or any subgroup of $GL(n, \mathbb{R})$. In this case

$$\mathrm{Ad}(A)(\xi) = A\xi A^{-1}, \quad \text{where } A \in GL(n, \mathbb{R}),\ \xi \in \mathfrak{gl}(\mathbb{R}).$$

Proposition 14.8 ([dC, page 41])**.** *Let* $\xi, \eta \in \mathfrak{g}$*. Then*

$$[\xi, \eta] = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp(t\xi))\eta.$$

We set $\mathrm{ad}(\xi)\eta = [\xi, \eta]$*. The map* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is called* adjoint representation *of the Lie algebra.*

Proof. We note that

$$\begin{aligned}\mathrm{Ad}(\exp(t\xi))\eta &= d(R_{-\exp(t\xi)})_{\exp(t\xi)} d(L_{\exp(t\xi)})_e \eta \\ &= d(R_{-\exp(t\xi)})_{\exp(t\xi)}(X^L_\eta(\exp(t\xi))) \\ &= \phi_t^* X^L_\eta(e)\end{aligned}$$

where $\phi_t = R_{\exp(t\xi)}$ is the flow of X^L_ξ. So

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp(t\xi))\eta = \frac{d}{dt}\Big|_{t=0} (\phi_t^* X^L_\eta)(e) = [X^L_\xi, X^L_\eta](e) = [\xi, \eta].$$

□

Example 14.9. Let $G = GL(n, \mathbb{R})$ or a subgroup. Then for $\xi, \eta \in \mathfrak{gl}(n, \mathbb{R})$

$$[\xi, \eta] = \frac{d}{dt}\Big|_{t=0} e^{t\xi} \eta e^{-t\xi} = \xi\eta - \eta\xi.$$

Continuous group actions

Definition 14.10. Let G be a group and M a set. Suppose that G acts on M on the left. For any $p \in M$:

- Let G_p denote the *stabilizer of* p, that is, $G_p = \{g \in G : g \cdot p = p\}$
- Let $G \cdot p$ denote the *orbit of* p, that is, $G \cdot p = \{g \cdot p : g \in G\}$.

We say G acts on M *freely* if $G_p = \{e\}$ for each $p \in M$. We say that G acts *transitively* if $M = G \cdot p$ for some $p \in M$ (which implies $M = G \cdot p$ for all $p \in M$).

Definition 14.11 (topological group)**.** We say that G is a *topological group* if G is a topological space together with a group structure such that the map $G \times G \to G$ given by $(x, y) \mapsto xy^{-1}$ is continuous.

Definition 14.12. Let G be a group and M a set, and suppose that G acts on M on the left. Let $\phi : G \times M \to M$ denote the action.

(i) If G is a topological group and M is a topological space, we say the action is *continuous* if ϕ is continuous as a map from the product space.
(ii) If G is a Lie group and M is smooth, then we say that the action is *smooth* if ϕ is smooth if ϕ is smooth as a map from the product manifold.

Lemma 14.13. *Let* G *be a group, let* M *be a topological space. Equip* G *with the discrete topology. Then* $\phi : G \times M \to M$ *is continuous if and only if for each* $g \in G$*, the map* $\phi_g : M \to M$ *is continuous.*

Proof. ($\Rightarrow$) If ϕ is continuous, then we note that $\phi_g = \phi \circ i_g$, where $i_g : M \to G \times M$ is the map $i_g(p) = (g, p)$, which is continuous, since G is given the discrete topology.

($\Leftarrow$) Suppose that each ϕ_g is continuous. Let U be an open subset of M. Then we note that

$$\phi^{-1}(U) = \bigcup_{g \in G} (\{g\} \times \phi_g^{-1}(U)).$$

Each of the sets in the union is open, and hence so is the union. □

Definition 14.14. Let G be a topological group and let M be a Hausdorff topological manifold. Suppose G acts on M on the left continuously. We say that the action is *proper* if for any compact $K \subset M$, the set $G_K := \{g \in G : \phi_g(K) \cap K \neq \varnothing\}$ is relative compact in G, i.e. the closure of G_K is compact. (This is automatic if G is compact.)

Example 14.15. Suppose that $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication. Then this action is not proper. On the other hand if $\mathbb{C}^*$ acts on $\mathbb{C}^*$, then the action is proper.

Remark 14.16.
- (i) Suppose that G is a discrete group. The action is continuous and proper if and only if for each compact subset $K \subset M$, the set G_K is finite. In particular, when $K = \{p\}$, $G_K = G_p$, we see that the stabilizer G_p of p is finite.
- (ii) Suppose that G is discrete. Suppose that the action is continuous, proper, and free. We already know that for any $p \in M$ there is an open neighborhood U of p in M such that $\overline{U}$ is compact. Because $G_{\overline{U}}$ is finite, we claim that $(G \cdot p) \cap U$ is finite. Because M is Hausdorff, there is an open neighborhood U' of p such that $U' \cap \phi_g(U') = \varnothing$ for each $g \in G \setminus \{e\}$. This means that the action is "properly discontinuous."

Example 14.17. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, a Lie group. Also $S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : |z_0|^2 + \cdots |z_n|^2 = 1\}$. Let S^1 act on S^{2n+1} by the rule

$$\lambda \cdot (z_0, \dots, z_n) = (\lambda z_0, \dots, \lambda z_n).$$

This action is smooth. The action is also proper because S^1 is compact. Moreover the action is free.

Theorem 14.18. *Let G be a Lie group and let M be a smooth manifold. If G acts on M smoothly, freely, and properly, then there is a unique smooth structure on M/G such that $\pi : M \to M/G$ is a smooth submersion.*

Example 14.19. Let $\pi : S^{2n+1} \to P_n(\mathbb{C}) = S^{2n+1}/S^1$ be the projection. We already constructed a C^∞ atlas on $P_n(\mathbb{C})$. We can check that π is a C^∞ submersion with respect to this C^∞ structure on $P_n(\mathbb{C})$. Theorem 14.18 implies that this C^∞ structure is unique with these properties. It follows that $P_n(\mathbb{C})$ is diffeomorphic to S^{2n+1}/S^1, where S^{2n+1}/S^1 is equipped with the unique C^∞ structure given by Theorem 14.18.

15. Wednesday, November 4, 2015

Definition 15.1 (Smooth fibration)**.** A map $\pi : E \to B$ is a *smooth fibration* with *total space* E, *base* B, and *fiber* F if

- (i) E, B, F are smooth manifolds.
- (ii) π is a surjective smooth map.
- (iii) There is an open cover $\{U_\alpha : \alpha \in I\}$ of B and smooth diffeomorphisms

$$h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$$

  such that $\pi|_{\pi^{-1}(U_\alpha)} = \mathrm{pr}_1 \circ h_\alpha$, where $\mathrm{pr}_1 : U_\alpha \times F \to U_\alpha$ is the projection to the first factor. (It follows that π is a submersion.)

Example 15.2. Take $E = B \times F$ with $\pi : E \to B$ being projection onto the first factor. This is called the *product fiber bundle* with base B and fiber F.

Definition 15.3. We say that $\pi : E \to B$ is a *trivial fiber bundle* over B with fiber F if there is a smooth diffeomorphism $h : E \to B \times F$ such that $\pi = \mathrm{pr}_1 \circ h$.

Example 15.4. If $\pi : E \to B$ is a smooth vector bundle of rank r, then $\pi : E \to B$ is a smooth fibration with fiber $\mathbb{R}^r$. But the converse is not true: the transition functions for a vector bundle need to satisfy some additional linearity requirement.

Example 15.5. A covering space is a smooth fibration with discrete fiber.

Theorem 15.6. *Let G be a Lie group and let M be a smooth manifold. If G acts on M smoothly, freely, and properly, then there is a unique smooth structure on M/G such that $\pi : M \to M/G$ is a smooth fibration with fiber G.*

Example 15.7. The map $\pi : S^{2n+1} \to P_n(\mathbb{C})$ is a smooth circle bundle, known as the *Hopf fibration*.

Riemannian submersions

Let $f : (M, g) \to (N, h)$ be a smooth submersion between Riemannian manifolds. For a point $p \in M$, let $q = f(p) \in N$. Then we have an exact sequence of the form

$$0 \to T_p f^{-1}(q) \to T_p M \stackrel{df_p}{\to} T_q N \to 0.$$

Let H_p be the orthogonal complement of $T_p f^{-1}(q)$ in $T_p M$ (using the metric $\langle -, - \rangle_p$). If we restrict df_p to H_p, then we see that $df_p|_{H_p}$ gives a linear isomorphism $H_p \cong T_q N$.

Definition 15.8 (Riemannian submersion)**.** We say that $f : (M, g) \to (N, h)$ is a *Riemannian submersion* if $df|_{H_p} : H_p \to T_{f(p)} N$ is an inner product space isomorphism. This means that for any $u, v \in H_p$, we have

$$\langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_q$$

Theorem 15.9. *If (M, g) is a Riemannian manifold and G is a Lie group acting smoothly, freely, and properly on M and in addition the action is by isometries, then there is a unique Riemannian metric $\hat{g}$ on M/G such that $\pi : (M, g) \to (M/G, \hat{g})$ is a Riemannian submersion.*

Proof. To determine this metric, we write

$$\hat{g}(q)(u, v) = g(p)((d\pi|_{H_p})^{-1} u, (d\pi|_{H_p})^{-1} v)$$

where $p \in \pi^{-1}(q)$. The right hand side is independent of choice of $p \in \pi^{-1}(q) = G \cdot p$ since $(d\phi_g)_p$ defines an isomoetry from H_p to $H_{g \cdot p}$. $\square$

Example 15.10. Use the round metric g_{can} on S^{2n+1} induced by the Euclidean metric on $\mathbb{R}^{2n+2}$. Then S^1 acts on S^{2n+1} smoothly, freely, properly, and isometrically. So there is a unique Riemannian metric $\hat{g}_{can}$ on $P_n(\mathbb{C})$ such that $\pi : S^{2n+1} \to P_n(\mathbb{C})$ is a Riemannian submersion. When $n = 1$, the space $P_n(\mathbb{C})$ is diffeomorphic to S^2 and $(P_1(\mathbb{C}), \hat{g}_{can})$ is isometric to $(S^2, \frac{1}{4} g_{can})$. (See Example 15.15 below.) So $\pi : S^3(1) \to S^2(\frac{1}{2})$ is a Riemannian submersion.

Theorem 15.11. *Let G be a Lie group and let H be a closed Lie subgroup. Then there is a unique smooth structure on G/H such that*

- *$\pi : G \to G/H$ is a smooth submersion and*
- *the action $\phi : G \times G/H \to G/H$ is smooth.*

Theorem 15.12. *If G is a Lie group and M is a smooth manifold, then the following are equivalent.*

(i) *G acts on M transitively, smoothly, and H is the stabilizer of some $p \in M$*
(ii) *M is diffeomorphic to G/H.*

Example 15.13. Let $\phi : SO(n+1) \times S^n \to S^n$ be the smooth map described by $(A, x) \mapsto Ax$. The action is smooth, transitive. The stabilizer of $(0, 0, \ldots, 0, 1)$ is

$$\left\{ \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} : A \in SO(n) \right\} \simeq SO(n).$$

So there is a map

$$SO(n+1)/SO(n) \to S^n$$
$$A \cdot SO(n) \mapsto A \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

which is a diffeomorphism.

By Assignment 7 (3), there is a bi-invariant metric g on $SO(n+1)$. There is a unique metric $\hat{g}$ on $SO(n+1)/SO(n)$ such that π is a Riemannian submersion.

Assignment 8 (2): $(SO(n+1)/SO(n), \hat{g})$ is isometric to $(S^n, \lambda g_{\text{can}})$ for some constant $\lambda > 0$.

Example 15.14. Let $Gr(k, n) = \{V \subset \mathbb{R}^n : V \ k\text{-dimensional subspace of } \mathbb{R}^n\}$. In particular we have $\mathbb{P}_n(\mathbb{R}) = \mathrm{Gr}(1, n+1)$. Note that $O(n)$ acts transitively on $\mathrm{Gr}(k, n)$ and the stabilizer of $\mathbb{R}^k \times \{0\}$ can be identified with $O(k) \times O(n-k)$. We may identify

$$\mathrm{Gr}(k, n) = O(n)/(O(k) \times O(n-k))$$

where the right hand side is a homogeneous space, which is a smooth manifold. The bi-invariant metric on $O(n)$ induces a Riemannian metric on $Gr(k, n)$, and $O(n)$ isometrically on $\mathrm{Gr}(k, n)$.

For example, we may write

$$\mathrm{Gr}(1, n+1) = \frac{O(n+1)}{O(1) \times O(n)} = \frac{1}{\{\pm 1\}} \frac{O(n+1)}{O(n)} = \frac{1}{\{\pm 1\}} \frac{SO(n+1)}{SO(n)} = \frac{1}{\{\pm 1\}} S^n.$$

Example 15.15. We have a diagram

$$\begin{array}{ccc} S^3 & \xrightarrow{\ \pi\ } & S^2 \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle j} \\ & & P_1(\mathbb{C}) \end{array}$$

where the diffeormophism $j^{-1} : P_1(\mathbb{C}) \to S^2$ is

$$[z_1, z_2] \mapsto \left(\frac{2 z_1 \bar{z}_2}{|z_1|^2 + |z_2|^2}, \frac{|z_2|^2 - |z_1|^2}{|z_1|^2 + |z_2|^2} \right)$$

and

$$\pi : S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\} \to S^2 = \{(w, z) \in \mathbb{C} \times \mathbb{R} : |w|^2 + z^2 = 1\}$$

is given by

$$(z_1, z_2) \mapsto (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2)$$

Let $\hat{g}_{\text{can}}$ be the unique metric on $P_1(\mathbb{C})$ such that $p : (S^3, g_{\text{can}}) \to (P_1(\mathbb{C}), \hat{g}_{can})$ is a Riemannian submersion. We want to compute $\hat{g} = j^*\hat{g}_{\text{can}}$.

Write

$$\begin{cases} z_1 = \sin\lambda e^{i\theta_1} \\ z_2 = \cos\lambda e^{i\theta_2} \end{cases}.$$

These coordinates cover almost all of S^3 and because metrics are continuous, this is sufficient for our purposes. On S^2 we use spherical coordinates

$$\begin{cases} x = \sin\phi\cos\theta \\ y = \sin\phi\sin\theta \\ z = \cos\phi \end{cases}.$$

We already know that $g_{can}^{S^2(1)} = d\phi^2 + (\sin^2\phi)d\theta^2$. If we write $z_j = x_j + iy_j$, then we note that

$$\begin{cases} x_1 = \sin\lambda\cos\theta_1 \\ y_1 = \sin\lambda\sin\theta_1 \\ x_2 = \cos\lambda\cos\theta_2 \\ y_2 = \cos\lambda\sin\theta_2 \end{cases}.$$

We compute that

$$g_{can}^{S^3(1)} = d\lambda^2 + \sin^2\lambda d\theta_1^2 + \cos^2\lambda d\theta_2^2.$$

In these coordinates, we find that

$$(\sin\lambda e^{i\theta_1}, \cos\lambda e^{i\theta_2}) \mapsto (\sin(2\lambda)e^{i(\theta_1-\theta_2)}, \cos^2\lambda - \sin^2\lambda).$$

In other words, $\phi = 2\lambda$ and $\theta = \theta_1 - \theta_2$. We find that

$$d\pi(\frac{\partial}{\partial\lambda}) = 2\frac{\partial}{\partial\phi}, \quad d\pi(\frac{\partial}{\partial\theta_1}) = \frac{\partial}{\partial\theta}, \quad d\pi(\frac{\partial}{\partial\theta_2}) = -\frac{\partial}{\partial\theta}.$$

We note that

$$\ker(d\pi) = \mathbb{R}(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}).$$

We find that the horizontal subspace is

$$H = (\ker d\pi)^\perp = \mathbb{R}\frac{\partial}{\partial\lambda} \oplus \mathbb{R}(\cos^2\lambda\frac{\partial}{\partial\theta_1} - \sin^2\lambda\frac{\partial}{\partial\theta_2}).$$

Let $\tilde{X}$ denote the horizontal lift of X. Then we find that

$$\widetilde{\frac{\partial}{\partial\phi}} = \frac{1}{2}\frac{\partial}{\partial\lambda}, \quad \widetilde{\frac{\partial}{\partial\theta}} = \cos^2\lambda\frac{\partial}{\partial\theta_1} - \sin^2\lambda\frac{\partial}{\partial\theta_2}.$$

We know that

$$\begin{aligned}
\hat{g}(\frac{\partial}{\partial\phi},\frac{\partial}{\partial\phi}) &= g_{\text{can}}^{S^3(1)}(\widetilde{\frac{\partial}{\partial\phi}},\widetilde{\frac{\partial}{\partial\phi}}) = g_{\text{can}}^{S^3(1)}(\frac{1}{2}\frac{\partial}{\partial\lambda},\frac{1}{2}\frac{\partial}{\partial\lambda}) = \frac{1}{4},\\
\hat{g}(\frac{\partial}{\partial\phi},\frac{\partial}{\partial\theta}) &= g_{\text{can}}^{S^3(1)}(\widetilde{\frac{\partial}{\partial\phi}},\widetilde{\frac{\partial}{\partial\theta}}) = 0\\
\hat{g}(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}) &= g_{\text{can}}^{S^3(1)}(\widetilde{\frac{\partial}{\partial\theta}},\widetilde{\frac{\partial}{\partial\theta}}) = \cos^4\lambda\sin^2\lambda+\sin^4\lambda\cos^2\lambda\\
&= \sin^2\lambda\cos^2\lambda = \frac{1}{4}\sin(2\lambda)^2 = \frac{1}{4}\sin^2\phi.
\end{aligned}$$

We see that

$$\hat{g} = \frac{1}{4}(d\phi^2+\sin^2\phi d\theta^2) = \frac{1}{4}g_{can}^{S^2(1)}.$$

16. Monday, November 9, 2015

Affine connections

Definition 16.1 (affine connection)**.** An *affine connection* ∇ on a smooth manifold M is a map

$$\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M), \quad (X,Y)\mapsto\nabla_X Y$$

such that for each $X,Y,Z\in\mathfrak{X}(M)$ and $f,g\in C^\infty(M)$, we have

(i) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$.

(ii) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$

(iii) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$.

Remark 16.2.

- In the above definition:
  i): for fixed $Y\in\mathfrak{X}(M)$, the map $X\mapsto\nabla_X Y$ is $C^\infty(M)$-linear.
  ii) and iii): for fixed $X\in\mathfrak{X}(M)$, the map $\nabla_X:\mathfrak{X}(M)\to\mathfrak{X}(M)$ is $\mathbb{R}$-linear, and satisfies the Leibniz rule.
- The Lie derivative $L:\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$, $(X,Y)\to L_X Y=[X,Y]$, is NOT an affine connection: it does not satisfy (i), although it satisfies (ii) and (iii).

Remark 16.3. If ∇_1 and ∇_2 are affine connections, then for $X\in\mathfrak{X}(M)$, the map

$$(\nabla_1)_X-(\nabla_2)_X:\mathfrak{X}(M)\to\mathfrak{X}(M)$$

is $C^\infty(M)$-linear and can be viewed as a section of $\text{End}(TM)$. That is, we may write

$$\nabla_1-\nabla_2\in C^\infty(M,T^*M\otimes T^*M\otimes TM)$$

The space of affine connections is an affine space associated to the vector space $C^\infty(M,T^1_2M)$.

We now study connections in local coordinates. Let (U,ϕ) be a chart for M and write $\phi=(x_1,\dots,x_n)$. The list $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$ form a smooth frame for $TM|_U=TU$. Then

$$\nabla_{\frac{\partial}{\partial x_i}}(\frac{\partial}{\partial x_j}) = \sum_k\Gamma_{ij}^k\frac{\partial}{\partial x_k}$$

for some $\Gamma_{ij}^k\in C^\infty(U)$.

If X and Y are smooth vector fields on U, we may write

$$X = \sum_i a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \sum_j b_j \frac{\partial}{\partial x_j}$$

where $a_i, b_j \in C^\infty(U)$. We find that

$$\nabla_X Y = \sum_{k=1}^n \left(\sum_{i=1}^n a_i \frac{\partial b_k}{\partial x_i} + \sum_{i,j=1}^n \Gamma_{ij}^k a_i b_j \right) \frac{\partial}{\partial x_k}.$$

Definition 16.4 (Vector field along a curve)**.** Let M be a smooth manifold and $c : I \to M$ a smooth curve. A smooth vector field along c is a smooth map $V : I \to TM$ such that $\pi \circ V = c$, that is, for each $t \in I$, we have $V(t) \in T_{c(t)}M$.

In local coordinates, if we restrict c to I' such that $c(I') \subset U$. Then

$$V(t) = \sum_{i=1}^n a_i(t) \frac{\partial}{\partial x_i}\Big|_{c(t)}$$

for $a_i \in C^\infty(I')$.

Example 16.5. The tangent vector field $\frac{dc}{dt}$ is a smooth vector field along c.

Proposition 16.6. *Let M be a smooth manifold with an affine connection ∇. Then there is a unique correspondence taking a smooth curve $c : I \to M$ together with a smooth vector field $V : I \to TM$ along c to a smooth vector field $\frac{DV}{dt} : I \to TM$ along c, called the* covariant derivative of V along c *such that*

(i) $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$
(ii) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$
(iii) *If $V = Y \circ c$ for some $Y \in \mathfrak{X}(M)$, then*

$$\frac{DV}{dt}(t) = \nabla_{\frac{dc}{dt}(t)} Y.$$

In local coordinates, consider the case $c : I \to U$, where (U, ϕ) is a local coordinate chart. Then $\phi \circ c : I \to \phi(U) \subset \mathbb{R}^n$ is given by $\phi \circ c(t) = (x_1(t), \ldots, x_n(t))$, where $x_i \in C^\infty(I)$. On U, we may write

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

And we may write

$$V(t) = \sum_{i=1}^n a_i(t) \frac{\partial}{\partial x_i}\Big|_{c(t)}, \quad \frac{dc}{dt}(t) = \sum_{i=1}^n \frac{dx_i}{dt}(t) \frac{\partial}{\partial x_i}\Big|_{c(t)}$$

Then

$$\begin{aligned} \frac{DV}{dt} &= \frac{D}{dt}\left(\sum_{i=1}^n a_i(t) \frac{\partial}{\partial x_i}\Big|_{c(t)} \right) \\ &= \sum_{i=1}^n \frac{D}{dt}(a_i(t) \frac{\partial}{\partial x_i}\Big|_{c(t)}) \\ &= \sum_{i=1}^n \frac{da_i}{\partial t}(t) \frac{\partial}{\partial x_i}\Big|_{(c(t))} + a_i \frac{D}{dt}(\frac{\partial}{\partial x_i}\Big|_{c(t)}) \end{aligned}$$

where

$$\frac{D}{dt}(\frac{\partial}{\partial x_i}|_{c(t)}) = \nabla_{\frac{dc}{dt}(t)}\frac{\partial}{\partial x_i} = \sum_{j=1}^{n}\frac{dx_j}{dt}(t)\nabla_{\frac{\partial}{\partial x_i}|_{c(t)}}\frac{\partial}{\partial x_i} = \sum_{j=1}^{n}\frac{dx_j}{dt}(t)\sum_{k=1}^{n}\Gamma_{ji}^{k}(c(t))\frac{\partial}{\partial x_k}\Big|_{c(t)}$$

Then we conclude that

$$\frac{DV}{dt} = \sum_{k=1}^{n}\left(\frac{da_k}{dt} + \sum_{i,j=1}^{n}\Gamma_{ij}^{k}\frac{dx_i}{dt}a_j\right)\frac{\partial}{\partial x_k}.$$

Parallel transport

Definition 16.7. Let M be a smooth manifold with an affine connection ∇. A smooth vector field V along smooth curve $c : I \to M$ is *parallel* if $\frac{DV}{dt}(t) = 0$ for all $t \in I$.

Proposition 16.8. *Let M be a smooth manifold with an affine connection ∇. Let $c : I \to M$ be a smooth curve and let $t_0 \in I$. For each tangent vector $V_0 \in T_{c(t_0)}M$ there is a unique parallel vector field $V(t)$ along $c(t)$ with $V(t_0) = V_0$. The vector field $V(t)$ is called the* parallel transport *of V_0 along c.*

Proof. We may assume that $c(I) \subset U$ where U is a coordinate chart. We may write

$$V_0 = \sum_i a_i \frac{\partial}{\partial x_i}|_{c(t_0)}$$

for some $a_i \in \mathbb{R}$. We want to solve $\frac{DV}{dt} = 0$ and $V(t_0) = V_0$. In terms of local coordinates, this means that, for $k = 1, \dots, n$,

$$\begin{cases} \dfrac{da_k}{dt} + \displaystyle\sum_{i,j=1}^{n}\Gamma_{ij}^{k}\frac{dx_i}{dt}a_j = 0 \\ a_k(t_0) = a_k \end{cases}$$

If we write

$$\vec{a}(t) = \begin{bmatrix} a_1(t) \\ \vdots \\ a_n(t) \end{bmatrix}, \quad \vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

and let $A(t) = (A_{kj}(t))$, where

$$A_{kj}(t) = -\sum_{i=1}^{n}\Gamma_{ij}^{k}(x_1(t), \dots, x_n(t))\frac{dx_i}{dt}(t)$$

Then these conditions are equivalent to

$$\begin{cases} \frac{d}{dt}\vec{a}(t) = A(t)\vec{a}(t) \\ \vec{a}(t_0) = \vec{a} \end{cases}.$$

So the proposition follows from the existence and uniqueness of solutions to first order ODE's. □

Example 16.9. On $\mathbb{R}^n$ we can take the trivial connection $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = 0$. Then the parallel vector fields are just constant along curves.

Riemannian Connection

Definition 16.10. An affine connection ∇ on a smooth manifold M is said to be *symmetric* if for any smooth vector fields $X, Y \in \mathfrak{X}(M)$, we have

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

In terms of local coordinates, this places the requirement that $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Definition 16.11. Let (M, g) be a Riemannian manifold with affine connection ∇. We say that ∇ is *compatible with the metric* g if for each $X, Y, Z \in \mathfrak{X}(M)$, we have

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

Theorem 16.12 (Levi-Civita)**.** *Given a Riemannian manifold (M, g), there is a unique affine connection ∇ on M such that*

(i) ∇ *is symmetric and*
(ii) ∇ *is compatible with g.*

This connection is known as the Riemannian connection *or the Levi-Civita connection on the Riemannian manifolds (M, g).*

Proof. For uniqueness, suppose that ∇ is an affine connection satisfying (i) and (ii). Then for any $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned} &X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &= g(\nabla_X Y + \nabla_Y X, Z) + g([X,Z],Y) + g([Y,Z],X) \\ &= g([X,Y] + 2\nabla_Y X, Z) + g([X,Z],Y) + g([Y,Z],X). \end{aligned}$$

It follows that

$$\begin{aligned} g(\nabla_Y X, Z) = \frac{1}{2}\,&(X(g(Y,Z)) + Y(g(Z,X))) - Z(g(X,Y)) \\ &- g([X,Z],Y) - g([Y,Z],X) - g([X,Y],Z). \end{aligned} \tag{16.1}$$

Since Z is arbitrary, Equation (16.1) uniquely determines $\nabla_Y X$.

For existence, one defines $\nabla_Y X$ by (16.1) and shows that this is an affine connection satisfying (i) and (ii). $\square$

In terms of local coordinates: in (16.1), let

$$X = \frac{\partial}{\partial x_j}, \quad Y = \frac{\partial}{\partial x_i}, \quad Z = \frac{\partial}{\partial x_k}.$$

We obtain

$$g(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}) = \frac{1}{2}\left(\frac{\partial}{\partial x_j} g_{ik} + \frac{\partial}{\partial x_i} g_{kj} - \frac{\partial}{\partial x_k} g_{ij}\right),$$

where $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{l=1}^{n} \Gamma^l_{ij} \frac{\partial}{\partial x_l}$, so

$$\sum_{l=1}^{n} \Gamma^l_{ij} g_{lk} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} g_{ik} + \frac{\partial}{\partial x_i} g_{kj} - \frac{\partial}{\partial x_k} g_{ij}\right)$$

and hence

$$\Gamma^l_{ij} = \frac{1}{2}\sum_{k=1}^{n} g^{lk}\left(\frac{\partial}{\partial x_j} g_{ik} + \frac{\partial}{\partial x_i} g_{kj} - \frac{\partial}{\partial x_k} g_{ij}\right)$$

where g^{lk} is the l,k entry of the inverse of g.

17. Wednesday, November 11, 2015

Recall that the Levi-Civita connection on a Riemannian manifold (M,g) is the unique affine connection which is symmetric and compatible with the Riemannian metric g.

Definition 17.1. Let ∇ be an affine connection on a smooth manifold M. The *torsion* of ∇ is defined to be

$$\begin{aligned} T_\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (X,Y) &\mapsto \nabla_X Y - \nabla_Y X - [X,Y]. \end{aligned}$$

It is straighforward to check that:

Lemma 17.2. (i) *T_∇ is antisymmetric: $T_\nabla(X,Y) = -T_\nabla(Y,X)$.*
(ii) *T_∇ is $C^\infty(M)$-bilinear.*
So $T_\nabla \in C^\infty(M, \Lambda^2 T^*M \otimes TM)$ is a (1,2)-tensor on M.

By definition, an affine connection ∇ is symmetric if and only of $T_\nabla = 0$. So the "symmetric" condition is also known as the "torsion free" condition.

Proposition 17.3. *Let (M,g) be a Riemannian manifold, and let ∇ be an affine connection on M compatible with the Riemannian metric g. If V,W are smooth vector fields along a smooth curve $c: I\to M$ then*

$$\frac{d}{dt}\langle V,W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle,$$

where $\langle\ ,\ \rangle$ is the inner product defined by g, and $\frac{D}{dt}$ is the covariant derivative along c determined by ∇. In particular, if V,W are parallel vector fields along c then $\langle V,W\rangle$ is a constant function on I.

We will see later that Proposition 17.3 is a special case of a more general result.

Example 17.4. Let $M=\mathbb{R}^n$ and let $g_0 = dx_1^2+\cdots+dx_n^2$. Since all the g_{ij}'s are constant, we find that $\Gamma_{ij}^k = 0$. This means that $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = 0$. This implies that if $X=\sum_i a_i\frac{\partial}{\partial x_i}$ and $Y=\sum_j b_j\frac{\partial}{\partial x_j}$, then we see that

$$\nabla_X Y = \sum_{i,j}\left(a_i\frac{\partial b_j}{\partial x_i}\right)\frac{\partial}{\partial x_j}.$$

Recall that

$$L_X Y = \sum_{i,j}\left(a_i\frac{\partial b_j}{\partial x_i} - b_i\frac{\partial a_j}{\partial x_i}\right)\frac{\partial}{\partial x_j} = \nabla_X Y - \nabla_Y X.$$

This shows that ∇ is indeed torsion free.

Example 17.5. Let S^2 be equipped with the round metric. Use spherical coordinates

$$\begin{cases} x = \sin\phi\cos\theta \\ y = \sin\phi\sin\theta \\ z = \cos\phi \end{cases}.$$

In these coordinates, we know that $g_{can} = d\phi^2 + \sin^2\phi d\theta^2$. Write $(x_1, x_2) = (\phi, \theta)$. In terms of these coordinates, we have

$$g = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\phi \end{bmatrix} \quad \text{and} \quad g^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\phi} \end{bmatrix}.$$

Let $g_{ij,k} = \frac{\partial}{\partial x_k} g_{ij}$. We compute the Christoffel symbols of the Levi-Civita connection to be

$$\begin{aligned}
&\Gamma_{11}^1 = 0 \\
&\Gamma_{11}^2 = 0 \\
&\Gamma_{12}^1 = \Gamma_{21}^1 = 0 \\
&\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{2} g^{22}(g_{22,1} + g_{12,2} - g_{12,2}) = \frac{1}{2}\frac{1}{\sin^2\phi} 2\sin\phi\cos\phi = \cot\phi \\
&\Gamma_{22}^1 = \frac{1}{2} g^{11}(2g_{21,2} - g_{22,1}) = -\sin\phi\cos\phi \\
&\Gamma_{22}^2 = 0.
\end{aligned}$$

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial\phi}} \frac{\partial}{\partial\phi} &= \Gamma_{11}^1 \frac{\partial}{\partial\phi} + \Gamma_{11}^2 \frac{\partial}{\partial\theta} = 0 \\
\nabla_{\frac{\partial}{\partial\phi}} \frac{\partial}{\partial\theta} &= \nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\phi} = \Gamma_{12}^1 \frac{\partial}{\partial\phi} + \Gamma_{12}^2 \frac{\partial}{\partial\theta} = \cot\phi \frac{\partial}{\partial\theta} \\
\nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta} &= \Gamma_{22}^1 \frac{\partial}{\partial\phi} + \Gamma_{22}^2 \frac{\partial}{\partial\theta} = -\sin\phi\cos\phi \frac{\partial}{\partial\phi}.
\end{aligned}$$

Parallel transport along a meridian $\theta = \theta_0$.

The vector field $\frac{\partial}{\partial\phi}$ is parallel along $\theta = \theta_0$ since $\nabla_{\frac{\partial}{\partial\phi}} \frac{\partial}{\partial\phi} = 0$. From Proposition 17.3, the vector field $\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}$ is also parallel along $\theta = \theta_0$ since it is perpendicular to $\frac{\partial}{\partial\phi}$ and of constant length 1. We now verify this directly:

$$\nabla_{\frac{\partial}{\partial\phi}} \left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right) = \frac{-\cos\phi}{\sin^2\phi}\frac{\partial}{\partial\theta} + \frac{1}{\sin\phi} \cdot \cot\phi \frac{\partial}{\partial\theta} = 0.$$

Any parallel vector field along a meridian $\theta = \theta_0$ is of the form

$$a\frac{\partial}{\partial\phi} + b \cdot \frac{1}{\sin\phi}\frac{\partial}{\partial\theta}$$

where $a, b \in \mathbb{R}$ are constants.

Parallel transport along a parallel $\phi = \phi_0$. Write $(x_1(\theta), x_2(\theta)) = (\phi_0, \theta)$. A vector field $V(\theta) = a_1(\theta)\frac{\partial}{\partial\phi} + a_2(\theta)\frac{\partial}{\partial\theta}$ along $\phi = \phi_0$ is parallel if and only if

$$\begin{cases} \dfrac{da_1}{d\theta} + \Gamma_{22}^1 a_2 = 0 \\ \dfrac{da_2}{d\theta} + \Gamma_{21}^2 a_1 = 0 \end{cases}$$

where $\Gamma_{22}^1 = -\sin\phi_0\cos\phi_0$, $\Gamma_{21}^2 = \cot\phi_0$. The above two equations can be rewritten as

$$\frac{d}{d\theta}\begin{bmatrix} a_1(\theta) \\ \sin\phi_0 a_2(\theta) \end{bmatrix} = \begin{bmatrix} 0 & \cos\phi_0 \\ -\cos\phi_0 & 0 \end{bmatrix}\begin{bmatrix} a_1(\theta) \\ \sin\phi_0 a_2(\theta) \end{bmatrix}$$

The solution is

$$\begin{bmatrix} a_1(\theta) \\ \sin\phi_0 a_2(\theta) \end{bmatrix} = \begin{bmatrix} \cos((\cos\phi_0)\theta) & \sin((\cos\phi_0)\theta) \\ -\sin((\cos\phi_0)\theta) & \cos((\cos\phi_0)\theta) \end{bmatrix} \begin{bmatrix} a_1(0) \\ \sin\phi_0 a_2(0) \end{bmatrix}$$

Let $a_1(0) = 1$ and $a_2(0) = 0$, we see that the parallel transport of the unit vector $\frac{\partial}{\partial\phi}$ along $\phi = \phi_0$ is

$$\cos((\cos\phi_0)\theta)\frac{\partial}{\partial\phi} - \frac{\sin((\cos\phi_0)\theta)}{\sin(\phi_0)}\frac{\partial}{\partial\theta}.$$

Let $a_1(0) = 0$ and $a_2(0) = \frac{1}{\sin\phi_0}$, we see that the parallel transport of the unit vector $\frac{1}{\sin\phi_0}\frac{\partial}{\partial\theta}$ along $\phi = \phi_0$ is

$$\sin((\cos\theta_0)\theta)\frac{\partial}{\partial\phi} + \frac{\cos((\cos\phi_0)\theta)}{\sin\phi_0}\frac{\partial}{\partial\theta}$$

Another way to see it is to consider a cone C tangent to S^2 along the circle $\phi = \phi_0$. Then for any p on the circle $\phi = \phi_0$, $T_pC = T_pS^2$. By Assignment 8 (4), the parallel tranport along $\phi = \phi_0$ defined by the Levi-Civita connection on C and the Levi-Civita connection on S^2 are the same. See page 79 of [GHL] for details.

Geodesics

Definition 17.6. Let M be a Riemannian manifold and let $\gamma : I \to M$ be a smooth curve. Then we say that γ is *geodesic* at $t_0 \in I$ if $\frac{D}{dt}(\frac{d\gamma}{dt})(t_0) = 0$, where we are using the Levi-Civita connection ∇. We say that γ is a *geodesic* if it is geodesic at each point of its domain.

By Proposition 17.3, if γ is a geodesic, then $|\frac{d\gamma}{dt}|$ is constant. Assume that $|\frac{d\gamma}{dt}| = c > 0$. We may parametrize by arc length to get $|\frac{d\gamma}{dt}| = 1$. In terms of local coordinates $\phi \circ \gamma(t) = (x_1(t), \ldots, x_n(t))$, we get the equation

$$\frac{d^2x_k}{dt^2} + \sum_{i,j}\Gamma^k_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt} = 0.$$

Example 17.7 (Euclidean space)**.** $M = \mathbb{R}^n$ equipped with the Euclidean metric $g_0 = dx_1^2 + \cdots + dx_n^2$. Then $\Gamma^k_{ij} = 0$. geodesic $\gamma : I \to \mathbb{R}^2$ satisfies $\frac{d^2x_k}{dt^2} = 0$ and hence $x_k(t) = a_k + b_k t$ for $a_k, b_k \in \mathbb{R}$. It follows that γ is affine linear in each coordinate. We conclude the following: for each $\vec{a} \in \mathbb{R}^n$ and $\vec{b} \in T_{\vec{a}}\mathbb{R}^n$, the line $\gamma(t) = \vec{a} + \vec{b}t$ is the unique geodesic such that $\gamma(0) = \vec{a}$ and $\gamma'(0) = \vec{b}$.

Example 17.8 (round sphere)**.** Geodesics in a round sphere are great circles. See Assignment 9 (2).

18. Monday, November 16, 2015

Proposition 18.1. *Let (M, g) be a Riemannian manifold. Let p be a point of M and $v \in T_pM$. Then*

- *(Existence) There is an open interval $I = (a, b)$, where $-\infty \le a < 0 < b \le +\infty$, and a geodesic $\gamma : I \to M$, such that $\gamma(0) = p$ and $\gamma'(0) = v$.*
- *(Uniqueness) If $\beta : I' \to M$ is another geodesic satisfying $\beta(0) = p$ and $\beta'(0) = v$ then $I' \subset I$ and $\beta = \gamma|_{I'}$.*

There is a reformulation using the notion of a geodesic field.

Geodesic field and geodesic flow

Definition 18.2. Given a smooth curve $\gamma : I \to M$, we define $\tilde{\gamma} : I \to TM$ by $\tilde{\gamma}(t) = (\gamma(t), \gamma'(t))$. Then $\tilde{\gamma}$ is a smooth curve in TM.

Any smooth curve $w : I \to TM$ is of the form $w(t) = (c(t), V(t))$, where $c : I \to M$ is a smooth curve in M and $V(t)$ is a smooth vector field along $c(t)$; w is equal to $\tilde{\gamma}$ for some *geodesic* $\gamma : I \to M$ if and only if

$$c'(t) = V(t), \quad \frac{DV}{dt}(t) = 0. \tag{18.1}$$

Suppose that $c(I)$ is contained in a coordinate neighborhood $U \subset M$. Then $w(I)$ is contained in $TU \subset TM$. $\phi : U \to \phi(U) \subset \mathbb{R}^n$ and $\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned}
\phi \circ c(t) &= (x_1(t), \ldots, x_n(t)), \\
V(t) &= \sum_{i=1}^n y_i(t) \frac{\partial}{\partial x_i}|_{c(t)}, \\
\tilde{\phi} \circ w(t) &= (x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_n(t)).
\end{aligned}$$

Then (18.1) is equivalent to the following system of $2n$ 1st order ODE's.

$$\frac{dx_k}{dt}(t) = y_k(t), \quad \frac{dy_k}{dt} = -\sum_{i,j} \Gamma^k_{ij}(x) y_i y_j, \quad k = 1, \ldots, n. \tag{18.2}$$

These are equations for the integral curve of the following smooth vector field on TU:

$$G = \sum_k y_k \frac{\partial}{\partial x_k} - \sum_{i,j,k} \Gamma^k_{ij}(x_1, \ldots, x_n) y_i y_j \frac{\partial}{\partial y_k}.$$

G is independent of choice of coordinates. We obtain a smooth vector field G on TM, known as the *geodesic field*. Proposition 18.1 follows from the existence and uniqueness of integral curves of $G \in \mathfrak{X}(TM)$.

Given $(p, v) \in TM$, where $p \in M$ and $v \in T_pM$, let $\gamma : I \to M$ be the unique geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$ in Proposition 18.1, and define $\tilde{\gamma} : I \to TM$ as in Definition 18.2. Then $\tilde{\gamma}(0) = (p, v)$ and $\tilde{\gamma}'(0) = G(p, v) \in T_{(p,v)}(TM)$.

Applying the existence/uniqueness theorem for flows of vector fields on TM, we find the following: for each $(p, v) \in TM$, where $p \in M$ and $v \in T_pM$, there is an open neighborhood U of (p, v) in TM, a positive number $\delta > 0$, and a smooth map

$$\phi : (-\delta, \delta) \times U \to TM$$

such that

$$\begin{cases} \frac{\partial \phi}{\partial t}(t, q, w) = G(\phi(t, q, w)) \\ \phi(0, q, w) = (q, w) \end{cases}$$

Let $\gamma = \pi \circ \phi : (-\delta, \delta) \times U \to M$. Then for a fixed $(q, w) \in U \subset TM$, we find that

$$\gamma_{q,w}(t) := \gamma(t, q, w) = \pi(\phi(t, q, w))$$

is a geodesic such that $\gamma_{q,w}(0) = q$ and $\frac{d\gamma_{q,w}}{dt}(0) = w$. For $t \in (-\delta, \delta)$, we get $\phi_t : U \to TM$, the flow of G, called the *geodesic flow*.

Example 18.3. When $(M,g) = (\mathbb{R}, dx^2)$, we can identify $T\mathbb{R}$ with $\mathbb{R}^2$ via the map $(x, y\frac{\partial}{\partial x}) \mapsto (x,y)$. Then we see that

$$G = y\frac{\partial}{\partial x}.$$

The flow $\phi_t : T\mathbb{R} \to T\mathbb{R}$ is given by

$$\phi_t(x,y) = (x+ty, y)$$

where $t \in \mathbb{R}$.

Example 18.4. More generally, when $(M,g) = (\mathbb{R}^n, g_0)$, we find that

$$G = \sum_i y_i \frac{\partial}{\partial x_i}$$

and $\phi_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is given by

$$\phi_t(x,y) = (x+ty, y),$$

where $x, y \in \mathbb{R}^n$.

Connections on vector bundles

Definition 18.5. Let M be a smooth manifold and let $\pi : E \to M$ be a smooth vector bundle of rank r. A **connection on** E is an $\mathbb{R}$-bilinear map $\nabla : \mathfrak{X}(M) \times C^\infty(M,E) \to C^\infty(M,E)$ written $(X,s) \mapsto \nabla_X s$ such that for any $X \in \mathfrak{X}(M)$, $s \in C^\infty(M,E)$, and $f \in C^\infty(M)$,

(i) $\nabla_{fX} s = f\nabla_X s$, i.e., ∇ is $\mathbb{C}^\infty(M)$-linear in the first factor;
(ii) $\nabla_X(fs) = X(f)s + f\nabla_X s$, i.e., for fixed $X \in \mathfrak{X}(M)$, the map $\nabla_X : C^\infty(M,E) \to C^\infty(M,E)$ sending s to $\nabla_X s$ satisfies the Leibniz rule.

Example 18.6. An affine connection on M is the same as a connection on TM.

We introduce the following notation. We denote by $\Omega^p(M,E)$ the space of E-valued p-forms on E, that is,

$$\Omega^p(M,E) = C^\infty(M, \Lambda^p T^*M \otimes E).$$

With this notation, Definition 18.5 can be reformulated as follows.

Definition 18.7. A connection on E is an $\mathbb{R}$-linear map $\nabla : \Omega^0(M,E) \to \Omega^1(M,E)$ written $s \mapsto \nabla s$ such that for each $f \in C^\infty(M)$ and each $s \in \Omega^0(M,E)$, we have

$$\nabla(fs) = df \otimes s + f\nabla s.$$

Lemma 18.8. *If ∇_1 and ∇_2 are connections on E, then $\nabla_1 - \nabla_2 : \Omega^0(M,E) \to \Omega^1(M,E)$ is $C^\infty(M)$-linear.*

Proof. For $f : M \to \mathbb{R}$ a smooth function and $s : M \to E$ a smooth section, we have

$$\begin{aligned}(\nabla_1 - \nabla_2)(fs) &= \nabla_1(fs) - \nabla_2(fs) \\ &= df \otimes s + f\nabla_1 s - df \otimes s - f\nabla_2 s \\ &= f(\nabla_1 - \nabla_2)s.\end{aligned}$$

□

It follows that $\phi := \nabla_1 - \nabla_2$ can be viewed as an element of $\Omega^1(M, \mathrm{End}E)$. The space of connections on E is an affine space whose associated vector space is $\Omega^1(M, \mathrm{End}E)$.

In general if E, F are smooth vector bundles and $\phi : C^\infty(M, E) \to C^\infty(M, F)$ is a $C^\infty(M)$-linear map, then we can view ϕ as an element of $C^\infty(M, E^* \otimes F)$:

$$\phi(s)(p) = \phi(p)s(p) \in F_p.$$

Now we want to express our connection in terms of local coordinates. Let (U, ϕ) be a chart for M and write $\phi = (x_1, \dots, x_n)$. We get a smooth frame $\{\frac{\partial}{\partial x_i}\}$ for the tangent bundle $TM|_U$. We may suppose that we have a trivialization $h : E|_U \to U \times \mathbb{R}^r$. We get a smooth frame $e_1, \dots, e_r$ for $E|_U$. On U, we have

$$\nabla_{\frac{\partial}{\partial x_i}} e_j = \sum_{k=1}^{r} \Gamma_{ij}^k e_k$$

for some $\Gamma_{ij}^k \in C^\infty(U)$. The element ∇e_j is an E-valued one-form on U and we note that

$$\nabla e_j = \sum_{i=1}^{n}\sum_{k=1}^{r} \Gamma_{ij}^k dx_i \otimes e_k = \sum_{k=1}^{r} \omega_j^k e_k$$

where $\omega_j^k = \sum_{i=1}^n \Gamma_{ij}^k dx_i$ are smooth 1-forms on U. To define the connection one-forms $\omega_j^k \in \Omega^1(U)$ we only need a trivialization of $E|_U$ but not $TM|_U$

$$\nabla e_j = \sum_{k=1}^{j} \omega_j^k e_k$$

where $\omega_j^k \in \Omega^1(U)$.

Let $\{U_\alpha : \alpha \in I\}$ be an open cover of M such that $h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$ are local trivializations. Let $\{e_{1,\alpha}, \dots, e_{r,\alpha}\}$ be a C^∞-frame of $E|_{U_\alpha}$, so that h_α^{-1} is given by $h_\alpha^{-1}(x, (v_1, \dots, v_r)) = (x, \sum_{i=1}^r v_i e_{i,\alpha}(x))$, where $x \in U_\alpha$ and $(v_1, \dots, v_r) \in \mathbb{R}^r$. On U_α, define $(\omega_\alpha)_j^k \in \Omega^1(U_\alpha)$ by

$$\nabla e_{j,\alpha} = \sum_{k=1}^{r} (\omega_\alpha)_j^k \otimes e_{k,\alpha}.$$

For a global smooth section $s \in C^\infty(M, E)$, we can expand s on U_α as

$$s = \sum_{j=1}^{r} s_\alpha^j e_{j,\alpha}$$

for some s_α^j in $C^\infty(U_\alpha)$. By Leibniz rule,

$$\nabla s = \sum_{j=1}^{r} ds_\alpha^j e_{j,\alpha} + \sum_{j=1}^{r} s_\alpha^j \nabla e_{j,\alpha} = \sum_{j=1}^{r} ds_\alpha^j e_{j,\alpha} + \sum_{j,k=1}^{r} s_\alpha^j (\omega_\alpha)_j^k e_{k,\alpha}.$$

On U_α, define $(\nabla s)_\alpha^j \in \Omega^1(U_\alpha)$ by

$$\nabla s = \sum_{j=1}^{r} (\nabla s)_\alpha^j e_{j,\alpha}.$$

We see that

$$(\nabla s)_\alpha^j = ds_\alpha^j + \sum_{k=1}^{r} (\omega_\alpha)_k^j s_\alpha^k,$$

or equivalently,

$$\begin{bmatrix}(\nabla s)^1_\alpha\\ \vdots \\ (\nabla s)^r_\alpha\end{bmatrix} = \begin{bmatrix}ds^1_\alpha\\ \vdots \\ ds^r_\alpha\end{bmatrix} + \begin{bmatrix}(\omega_\alpha)^1_1 & \cdots & (\omega_\alpha)^1_r\\ \vdots & \ddots & \vdots\\ (\omega_\alpha)^r_1 & \cdots & (\omega_\alpha)^r_r\end{bmatrix}\begin{bmatrix}s^1_\alpha\\ \vdots \\ s^r_\alpha\end{bmatrix}. \tag{18.3}$$

We define

$$s_\alpha := \begin{bmatrix}s^1_\alpha\\ \vdots \\ s^r_\alpha\end{bmatrix} \in C^\infty(U_\alpha, \mathbb{R}^r), \quad (\nabla s)_\alpha := \begin{bmatrix}(\nabla s)^1_\alpha\\ \vdots \\ (\nabla s)^r_\alpha\end{bmatrix} \in \Omega^1(U_\alpha, \mathbb{R}^r), \tag{18.4}$$

and define a matrix-valued 1-form

$$\omega_\alpha := \begin{bmatrix}(\omega_\alpha)^1_1 & \cdots & (\omega_\alpha)^1_r\\ \vdots & \ddots & \vdots\\ (\omega_\alpha)^r_1 & \cdots & (\omega_\alpha)^r_r\end{bmatrix} \in \Omega^1(U_\alpha, \mathfrak{gl}(r,\mathbb{R})). \tag{18.5}$$

Then (18.3) can be written as

$$(\nabla s)_\alpha = ds_\alpha + \omega_\alpha s_\alpha$$

where $(\nabla s)_\alpha$ and ds_α are column vectors with components that are 1-forms, ω_α is a matrix with entries that are 1-forms, and s_α is a column vector with components that are smooth functions.

19. Wednesday, November 18, 2015

Let $\pi : E \to M$ be a smooth vector bundle of rank r over a smooth manifold M. Suppose that $\{U_\alpha : \alpha \in I\}$ is an open cover of M and $h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$ are local trivializations. The local trivialization h_α gives a smooth frame $\{e_{i,\alpha} : i = 1, \dots, r\}$ for $E|_{U_\alpha}$ such that $h_\alpha^{-1}(x, \vec{v}) = (x, \sum_{i=1}^r v_i e_{i,\alpha}(x))$. When $U_\alpha \cap U_\beta$ is nonempty, we also have transition functions

$$h_{\alpha\beta} = h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r, \quad (x, v) \mapsto (x, t_{\alpha\beta}(x)v)$$

where $t_{\alpha\beta}$ is a smooth map from $U_\alpha \cap U_\beta$ to $GL(r,\mathbb{R})$. Then $t_{\alpha\alpha}(x) = I_r$ for all $x \in U_\alpha$, where I_r is the $r \times r$ identity matrix, and $t_{\alpha\beta}(x)t_{\beta\gamma}(x)t_{\gamma\alpha}(x) = I_r$ for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

Conversely, given an open cover $\{U_\alpha : \alpha \in I\}$ of M and smooth maps $t_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r,\mathbb{R})$ satisfying $t_{\alpha\alpha}(x) = I_r$ for all $x \in U_\alpha$ and $t_{\alpha\beta}(x)t_{\beta\gamma}(x)t_{\gamma\alpha}(x) = I_r$ for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$, we may construct a smooth rank r vector bundle E over M by gluing the rank r product vector bundles $\{U_\alpha \times \mathbb{R}^r \to U_\alpha : \alpha \in I\}$ along $(U_\alpha \cap U_\beta) \times \mathbb{R}^r$ using $t_{\alpha\beta}$.

Let $s \in C^\infty(M, E)$ be a global section, and let $s_\alpha \in C^\infty(U_\alpha, \mathbb{R}^r)$ be defined as the previous lecture. Then $h_\alpha(x) = (x, s_\alpha(x))$ for $x \in U_\alpha$. On $U_\alpha \cap U_\beta$,

$$(x, s_\alpha(x)) = h_\alpha(x) = h_\alpha \circ h_\beta^{-1} \circ h_\beta(x) = h_\alpha \circ h_\beta^{-1}(x, s_\beta(x)) = (x, t_{\alpha\beta}(x)s_\beta(x)).$$

So we have

$$s_\alpha = t_{\alpha\beta} s_\beta. \tag{19.1}$$

In a similar fashion, let $(\nabla s)_\alpha \in \Omega^1(U_\alpha, \mathbb{R}^r)$ be defined as in the previous lecture. The

$$(\nabla s)_\alpha = t_{\alpha\beta}(\nabla s)_\beta. \tag{19.2}$$

The left hand side of (19.2) is

$$ds_\alpha + \omega_\alpha s_\alpha = d(t_{\alpha\beta}s_\beta) + \omega_\alpha t_{\alpha\beta}s_\beta = (dt_{\alpha\beta})s_\beta + t_{\alpha\beta}(ds_\beta) + \omega_\alpha t_{\alpha\beta}s_\beta$$

and the right hand side of (19.2) is

$$t_{\alpha\beta}ds_\beta + t_{\alpha\beta}\omega_\beta s_\beta.$$

Therefore,

$$\omega_\beta = t_{\alpha\beta}^{-1}dt_{\alpha\beta} + t_{\alpha\beta}^{-1}\omega_\alpha t_{\alpha\beta} \tag{19.3}$$

on $U_\alpha \cap U_\beta$. A connection $\nabla : \Omega^0(M,E) \to \Omega^1(M,E)$ is equivalent to a collection $\{\omega_\alpha \in \Omega^1(U_\alpha, \mathfrak{gl}(r,\mathbb{R}))\}$ satisfying (19.3) on $U_\alpha \cap U_\beta$.

Pullback bundle

Let $f : M \to N$ be a smooth map between smooth manifolds, and let $\pi : E \to N$ be a smooth vector bundle on N. Then we can define a bundle $\tilde{\pi} : f^*E \to M$ called the *pullback bundle* in the following manner. As a set

$$f^*E = \bigcup_{p\in M} E_{f(p)} = \{(p,q) \in M \times E : f(p) = \pi(p)\}.$$

The smooth structure is determined in the following manner. If $s : N \to E$ is a smooth section of E, then $f^*s : M \to f^*E$ given by

$$f^*s(p) = s(f(p)) \in E_{f(p)} =: (f^*E)_p$$

is a smooth section of f^*E. If $e_1, \ldots, e_r$ are a smooth frame for $E|_U$, where U is an open set in N, then $f^*e_1, \ldots, f^*e_r$ are a smooth frame of $f^*E|_{f^{-1}(U)}$. A section $s : f^{-1}(U) \to f^*E|_{f^{-1}(U)}$ is smooth if and only if we can write

$$s = \sum_{j=1}^{r} a_j f^*e_j$$

where the a_j are smooth functions on $f^{-1}(U)$. We have a pullback map

$$f^* : C^\infty(N,E) \to C^\infty(M, f^*E).$$

Suppose that $\{U_\alpha : \alpha \in I\}$ is an open cover of N with local trivializations $h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$, and define transition functions $t_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r,\mathbb{R})$ as before. Then

$$f^*t_{\alpha\beta} := t_{\alpha\beta} \circ f : f^{-1}(U_\alpha \cap U_\beta) = f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to GL(r,\mathbb{R})$$

are the transition functions of f^*E.

Definition 19.1 (pullback connection)**.** Let $f : M \to N$ be a smooth map between smooth manifolds, and let $\pi : E \to N$ be a smooth vector bundle together with a connection ∇. Then there is a unique connection $f^*\nabla$ on f^*E, called the *pullback connection*, such that

$$(f^*\nabla)(f^*s) = f^*(\nabla s)$$

for a smooth section $s : N \to E$.

In other words, if $s : N \to E$ is a smooth section, p is a point of M, and $X \in T_pM$, then

$$(f^*\nabla)_X(f^*s) = f^*(\nabla_{df_p(X)}s).$$

In terms of local trivializations, we know that if $e_1, \dots, e_r$ are a smooth frame of $E|_U$, then $f^*e_1, \dots, f^*e_r$ are a smooth frame for $f^*E|_{f^{-1}(U)}$. On U, we know that

$$\nabla e_j = \sum_{k=1}^{r} \omega_j^k \otimes e_k.$$

Then

$$(f^*\nabla)(f^*e_j) = f^*(\nabla e_j) = \sum_{k=1}^{r} f^*\omega_j^k \otimes f^*e_k.$$

Therefore, if $\{\omega_\alpha \in \Omega^1(U_\alpha, \mathfrak{gl}(r,\mathbb{R})) : \alpha \in I\}$ are connection 1-forms of the connection ∇ on $E \to N$, then $\{f^*\omega_\alpha \in \Omega^1(f^{-1}(U_\alpha), \mathfrak{gl}(r,\mathbb{R})) : \alpha \in I\}$ are connection 1-forms of the pullback connection $f^*\nabla$ on $f^*E \to M$.

We next consider the special case $E = TN$.

Definition 19.2. Let $F : M \to N$ be a smooth map between smooth manifolds. Define a pushforward map

$$F_* : \mathfrak{X}(M) = C^\infty(M, TM) \to C^\infty(M, F^*TN)$$

by

$$(F_*X)(p) = (dF_p)(X(p)) \in T_{F(p)}N = (F^*TN)_p,$$

and define a pullback map

$$F^* : \mathfrak{X}(N) = C^\infty(N, TN) \to C^\infty(M, F^*TN)$$

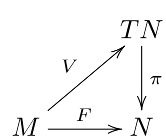
by

$$(F^*Y)(p) = Y(F(p)) \in T_{F(p)}N = (F^*TN)_p$$

Remark 19.3. Let $X \in \mathfrak{X}(M)$ be a smooth vector field on M, and let $Y \in \mathfrak{X}(N)$ be a smooth vector field on N. Then X and Y are F-related in the sense of Definition 13.10 if and only of

$$F_*X = F^*Y \in C^\infty(M, F^*TN).$$

Definition 19.4. An element in $C^\infty(M, F^*TN)$ is a smooth map $V : M \to F^*TN$ is such that the diagram



commutes. Following [dC], we call V a *smooth vector field along* $F : M \to N$.

As special cases of the above definition:

- In [dC, Chapter 2], we consider vector fields along a parametrized curve $\gamma : I \to N$, where I is an open interval in $\mathbb{R}$ and γ is a smooth map.
- In [dC, Chapter 3], we consider vector fields along a parametrized surface $s : A \to N$, where A is an open set in $\mathbb{R}^2$ and s is a smooth map.

Proposition 19.5. *Suppose that we have a smooth map $F : M \to N$ from a smooth manifold M to a Riemannian manifold (N, h), so that we have a pushforward map $F_* : \mathfrak{X}(M) \to C^\infty(M, f^*TN)$. Let ∇ be an affine connection on N, and let $D := F^*\nabla$ be the pull-back connection on F^*TN.*

(i) *If ∇ is compatible with the Riemannian metric h then*

(19.4) $X\langle V,W\rangle = \langle D_X V, W\rangle + \langle V, D_X W\rangle \ \forall X \in \mathfrak{X}(M)\ \forall V,W \in C^\infty(M, F^*TN).$

Here the inner product $\langle\ ,\ \rangle$ is defined by h.

(ii) *If ∇ is symmetric then*

(19.5) $$D_X F_* Y - D_Y F_* X = F_*([X,Y]) \quad \forall X, Y \in \mathfrak{X}(M).$$

In particular, if ∇ is the Levi-Civita connection then the pullback connection D satisfies (19.4) *and* (19.5).

Proof. Assignment 10 (1). □

Let N be a smooth manifold with an affine connection ∇.

Let $\gamma : I \to N$ be a smooth curve in N, and let V be a smooth vector field along γ. The covariant derivative along γ is given by

$$\frac{DV}{dt} = (\gamma^*\nabla)_{\frac{\partial}{\partial t}} V.$$

The following proposition, which is the same as Proposition 17.3, is a special case of part (i) of Proposition 19.5.

Proposition 19.6. *If ∇ is compatible with a Riemannian metric h on N then the covariant derivative along a parametrize curve $\gamma : I \to N$ satisfies*

$$\frac{d}{dt}\langle V,W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle$$

for any vector fields V, W along γ, where the inner product $\langle\ ,\ \rangle$ is defined by h.

Let $s : A \to N$ be a parametrized surface in N, where A is an open set in $\mathbb{R}^2$. Let (u,v) be coordinates on $\mathbb{R}^2$. Then $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$ is a smooth frame for TA. Let

$$\frac{\partial s}{\partial u} := s_* \frac{\partial}{\partial u},\ \frac{\partial s}{\partial v} := s_* \frac{\partial}{\partial v} \in C^\infty(A, s^*TN).$$

Let W be a vector field along this parametrized surface, that is, $W \in C^\infty(A, s^*TN)$. Then we define

$$\frac{DW}{\partial u} := (s^*\nabla)_{\frac{\partial}{\partial u}} W,\ \frac{DW}{\partial v} := (s^*\nabla)_{\frac{\partial}{\partial v}} W \in C^\infty(A, s^*TN).$$

Proposition 19.7. *If ∇ is symmetric then the covariant derivative along the parametrized surface $s : A \to N$ satisfies*

$$\frac{D}{\partial v}\frac{\partial s}{\partial u} = \frac{D}{\partial u}\frac{\partial s}{\partial v}.$$

Proof. Let $D := s^*\nabla$ be the pullback connection on s^*TN. Then

$$\frac{D}{\partial v}\frac{\partial s}{\partial u} - \frac{D}{\partial u}\frac{\partial s}{\partial v} = D_{\frac{\partial}{\partial v}}\big(s_*\frac{\partial}{\partial u}\big) - D_{\frac{\partial}{\partial u}}\big(s_*\frac{\partial}{\partial v}\big) = s_*([\frac{\partial}{\partial v}, \frac{\partial}{\partial u}]) = 0$$

where the second equality follows from part (ii) of Proposition 19.5. □

We now study the homogeneity of the geodesics. Let

$$\phi : (-\delta, \delta) \times U \to TM$$

be the geodesic flow defined on some open subset $U \subset TM$. Let $\gamma = \pi \circ \phi : (-\delta,\delta) \times U \to M$. Then $\phi(t,q,v) = (\gamma(t,q,v), \frac{\partial \gamma}{\partial t}(t,q,v))$.

Lemma 19.8. *If the map $\gamma(t,q,v)$ is defined for $t \in (-\delta,\delta)$, then for each $a>0$, the map $\gamma(t,q,av)$ is defined for $t \in (-\delta/a, \delta/a)$ and $\gamma(t,q,av) = \gamma(at,q,v)$.*

Proof. Observe that, if $\beta : (-\delta,\delta) \to M$ is a geodesic with $\beta(0) = q \in M$ and $\beta'(0) = v \in T_qM$, then $\tilde{\beta} : (-\delta/a, \delta/a) \to M$ defined by $\tilde{\beta}(t) = \beta(at)$ is a geodesic with $\tilde{\beta}(t) = q$ and $\tilde{\beta}'(0) = av$. □

Remark 19.9. If M is compact, the tangent bundle TM is not compact, so the flow may not exist for all time t. However, we can consider the sphere bundle $S(TM) = \{(x,v) \in TM : |v| = 1\}$, which is compact. The geodesic field G on TM is tangent to $S(TM)$, so it restricts to a vector field $\tilde{G}$ on $S(TM)$. By Lemma 7.8, the flow of $\tilde{G}$ is defined on $\mathbb{R} \times S(TM)$: $\tilde{\phi} : \mathbb{R} \times S(TM) \to S(TM)$. By the above Lemma 19.8, the geodesic flow ϕ is defined on $\mathbb{R} \times TM$.

20. Monday, November 23, 2015

Given $p \in M$, there is an open neighborhood V of p in M, an $\epsilon > 0$ and a $\delta > 0$ such that $\gamma(t,q,v)$ is defined for $-\delta < t < \delta$, $q \in V$, and $|v| < \epsilon$. By Lemma 19.8, $\gamma(t,q,v)$ is defined for $-2 < t < 2$, $q \in V$, and $|v| < \epsilon\delta/2$. So for any $p \in M$, there is an open neighborhood V of p in M and an $\epsilon > 0$ such that $\gamma(t,q,v)$ is defined for $-2 < t < 2$, $q \in V$, and $|v| < \epsilon$.

Definition 20.1 (Exponential Map)**.** Let $U_{(V,\epsilon)} = \{(q,w) \in TM : q \in V, |w| < \epsilon\}$. Define

$$\exp : U_{(V,\epsilon)} \longrightarrow M, \quad \exp(q,w) = \gamma(1,q,w).$$

Also define

$$\exp_p : B_\epsilon(0) \longrightarrow M, \quad \exp_p(v) = \gamma(1,p,v),$$

where $B_\epsilon(0) \subset T_pM$ is the open ball with center at the origin and with radius $\epsilon > 0$. (Geometrically, this means that we find the unique geodesic passing through p with velocity v and we flow for unit amount of time.)

Lemma 20.2. *The map $(d\exp_p)_0 : T_0(T_pM) = T_pM \to T_pM$ is the identity map.*

Proof.

$$(d\exp_p)_0(v) = \frac{d}{dt}\Big|_{t=0} \exp_p(tv) = \frac{d}{dt}\Big|_{t=0} \gamma(1,p,tv) = \frac{d}{dt}\Big|_{t=0} \gamma(t,p,v) = v.$$

□

Corollary 20.3. *There is an open neighborhood U of 0 in T_pM such that $\exp_p : U \to V := \exp_p(U)$ is a diffeomorphism.*

Definition 20.4. In Corollary 20.3, the open neighborhood V is called a *normal neighborhood of p in M*. If $\overline{B_\epsilon(0)} \subset U$, then $B_\epsilon(p) := \exp_p(B_\epsilon(0)) \subset M$ is called a *normal ball* (or *geodesic ball*) of radius $\epsilon > 0$ centered at p. The boundary $S_\epsilon(p) = \partial B_\epsilon(p)$ of this geodesic ball is called the *normal sphere* (or *geodesic sphere*) of radius $\epsilon > 0$ centered at p.

Example 20.5. The exponential $\exp_p : T_p\mathbb{R}^n \to \mathbb{R}^n$ is given by $\exp_p(v) = p + v$, which is a global diffeomorphism.

Example 20.6. The map $\exp_p : T_pS^n \to S^n$ is given by

$$\exp_p(v) = \begin{cases} p, & v = 0, \\ \cos(|v|)p + \sin(|v|)\frac{v}{|v|}, & v \neq 0 \end{cases}$$

This is a diffeomorphism of $B_\pi(0)$ onto $S^n \setminus \{-p\}$.

Minimizing properties of geodesics

Lemma 20.7 (Gauss)**.** *Let $p \in M$ and $v \in T_pM$ such that $\exp_p(v)$ is defined. Identify T_pM with $T_v(T_pM)$. Then for $w \in T_pM$, we have*

$$\langle (d\exp_p)_v(v), (d\exp_p)_v(w)\rangle = \langle v, w\rangle.$$

Proof. There exist $\delta, \epsilon > 0$ small enough such that $f(s,t) := \exp_p(t(v+sw))$ is defined for $t \in (-\delta, 1+\delta)$ and $s \in (-\epsilon, \epsilon)$. For any $s \in (-\epsilon, \epsilon)$, the curve $f_s : (-\delta, 1+\delta) \to M$ defined by $f_s(t) := f(s,t) = \exp_p(t(v+sw))$ is a geodesic with $f_s(0) = p$ and $f_s'(0) = v + sw$. So we have

$$\frac{D}{\partial t}\frac{\partial f}{\partial t}(s,t) = \frac{D}{dt} f_s'(t) = 0 \tag{20.1}$$

and $|\frac{\partial f}{\partial t}(s,t)| = |f_s'(t)| = |f_s'(0)| = |v + sw| \Rightarrow$

$$\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle(s,t) = |v+sw|^2 = |v|^2 + 2s\langle v, w\rangle + s^2|w|^2. \tag{20.2}$$

We also have

$$\begin{aligned} \frac{\partial f}{\partial s}(s,t) = (d\exp_p)_{t(v+sw)}(tw) \quad &\Rightarrow \quad \frac{\partial f}{\partial s}(0,t) = (d\exp_p)_{tv}(tw); \\ \frac{\partial f}{\partial t}(s,t) = (d\exp_p)_{t(v+sw)}(v+sw) \quad &\Rightarrow \quad \frac{\partial f}{\partial t}(0,t) = (d\exp_p)_{tv}(v). \end{aligned}$$

So

$$\begin{aligned} \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\rangle(0,1) &= \langle (d\exp_p)_v(v), (d\exp_p)_v(w)\rangle, \\ \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\rangle(0,0) &= 0. \end{aligned}$$

$$\begin{aligned} &\langle (d\exp_p)_v(v), (d\exp_p)_v(w)\rangle \\ &= \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\rangle(0,1) - \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\rangle(0,0) = \int_0^1 \frac{\partial}{\partial t}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\rangle(0,t)dt. \end{aligned}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\rangle = \langle \frac{D}{dt}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\rangle + \langle \frac{\partial f}{\partial t}, \frac{D}{\partial t}\frac{\partial f}{\partial s}\rangle \\ &= \langle \frac{\partial f}{\partial t}, \frac{D}{\partial t}\frac{\partial f}{\partial s}\rangle = \langle \frac{\partial f}{\partial t}, \frac{D}{\partial s}\frac{\partial f}{\partial t}\rangle = \frac{1}{2}\frac{\partial}{\partial s}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle \\ &= \frac{1}{2}\frac{\partial}{\partial s}(|v|^2 + 2s\langle v, w\rangle + s^2|w|^2) \\ &= \langle v, w\rangle + s|w|^2. \end{aligned}$$

The first equality follows from part (i) of Proposition 19.5; the second equality follows from (20.1); the third equality follows from part (ii) of Proposition 19.5; the fourth equality follows from part (i) of Proposition 19.5; the fifth equality follows from (20.2).

$$\frac{\partial}{\partial t}\langle\frac{\partial f}{\partial t},\frac{\partial f}{\partial s}\rangle(0,t)=\langle v,w\rangle \Rightarrow \langle(d\exp_p)_v(v),(d\exp_p)_v(w)\rangle=\int_0^1\langle v,w\rangle dt=\langle v,w\rangle.$$

□

Proposition 20.8. *Let (M,g) be a Riemannian manifold, $p\in M$, and U a normal neighborhood of p. Let $B\subset U$ be a normal ball with center p, that is, $B=\exp_p(B_\delta(0))$ for some $\delta>0$. Suppose that $\gamma:[0,1]\to B$ is a geodesic segment such that $\gamma(0)=p$ and $\gamma(1)=q$. Let $c:[0,1]\to M$ be a piecewise smooth curve such that $c(0)=p$ and $c(1)=q$. Then $l(\gamma)\le l(c)$, with equality if and only if the curves c and γ have the same image.*

Proof. We may assume that $c([0,1])\subset B$, since $l(c)\ge l(c|_{[0,t_1]})$ where $c(t_1)\in\partial B$ and $c(t)\subset B$ for $0\le t<t_1$. We may also assume that $c(t)\neq p$ for $t>0$, otherwise consider $c|_{[t_2,1]}$ where $c(t_2)=p$ and $c(t)\neq p$ for $t_2<t\le 1$.

Define $b:[0,1]\to B_\delta(0)\subset T_pM$ by $b(t)=\exp_p^{-1}(c(t))$. Then $b:[0,1]\to T_pM$ is a piecewise smooth curve in T_pM, and $c(t)=\exp_p(b(t))$. Since $c(t)\neq p$ for $t>0$, $b(t)\neq 0$ for $t>0$, so for $t\in(0,1]$ we may write

$$b(t)=r(t)v(t)$$

where $r(t)=|b(t)|>0$ and $v(t)=b(t)/|b(t)|$ are piecewise smooth. We have

$$\langle v(t),v(t)\rangle=1,\quad \langle v(t),v'(t)\rangle=0.$$

$$\frac{dc}{dt}(t)=(d\exp_p)_{b(t)}(b'(t))=r'(t)(d\exp_p)_{b(t)}(v(t))+r(t)(d\exp_p)_{b(t)}(v'(t)).$$

Therefore

$$\begin{aligned}\left|\frac{dc}{dt}(t)\right|^2 &= r'(t)^2|(d\exp_p)_{(b(t))}(v(t))|^2+r(t)^2|(d\exp_p)_{b(t)}(v'(t))|^2\\ &\quad +2r'(t)r(t)\langle(d\exp_p)_{(b(t))}(v(t)),(d\exp_p)_{(b(t))}(v'(t))\rangle\end{aligned}$$

Note that $v(t)$ is a scalar multiple of $b(t)$, so by Gauss's lemma.

$$\begin{aligned}&|(d\exp_p)_{(b(t))}(v(t))|^2=|v(t)|^2=1,\\ &\langle(d\exp_p)_{(b(t))}(v(t)),(d\exp_p)_{(b(t))}(v'(t))\rangle=\langle v(t),v'(t)\rangle=0.\end{aligned}$$

Therefore,

$$\left|\frac{dc}{dt}(t)\right|=\sqrt{r'(t)^2+r(t)^2|(d\exp_p)_{b(t)}(v'(t))|^2}\ge|r'(t)|\ge r'(t).$$

So the length of c satisfies

$$l(c)=\int_0^1\left|\frac{dc}{dt}(t)\right|dt\ge\int_0^1 r'(t)dt=r(1)-r(0)=l(\gamma).$$

Equality holds if and only if $v'(t)=0$ and $\frac{dr}{dt}\ge 0$. In this case, $v(t)=v$ is a constant unit vector, and

$$c(t)=\exp_p(r(t)v)$$

which has the same image as $\gamma(t)=\exp_p(l(\gamma)tv)$. □

21. Wednesday, November 25, 2015

Theorem 21.1. *Let (M, g) be a Riemannian manifold and let p be a point of M. Then there is an open neighborhood W of p in M and $\delta > 0$ such that for any $q \in W$, $\exp_q$ is a diffeomorphism from $B_\delta(0) \subset T_qM$ onto the geodesic ball $B_\delta(q)$, and $W \subset B_\delta(q)$.*

In particular, W is a normal neighborhood of q for any $q \in W$. We call W a totally geodesic neighborhood *of p in M.*

Proof. There is an open neighborhood V of p in M and an $\epsilon > 0$ such that $\gamma(t, q, v)$ is defined for any $t \in (-2, 2)$, $q \in V$, and $|v| < \epsilon$. Then $\exp_q(v) = \gamma(1, q, v)$ is defined for $(q, v) \in U_{(V,\epsilon)} := \{(q, v) \in TM : q \in V, |v| < \epsilon\}$.

Define $F : U_{(V,\epsilon)} \to M \times M$ be

$$F(q, v) = (q, \exp_q(v)).$$

We now compute

$$dF_{(p,0)} : T_{(p,0)}TM = T_pM \times T_pM \longrightarrow T_{(p,p)}(M \times M) = T_pM \times T_pM.$$

For any $q \in V$, we have $F(q, 0) = (q, \exp_q(0)) = (q, q)$. This implies that

$$dF_{(p,0)}(u, 0) = (u, u).$$

For any $v \in T_qM$, we have $F(p, v) = (p, \exp_p v)$. This implies that

$$dF_{(p,0)}(0, v) = (0, (d\exp_p)_0(v)) = (0, v).$$

Therefore

$$dF_{(p,0)} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$$

where $I : T_pM \to T_pM$ is the identity map. In particular, $dF_{(p,0)}$ is a linear isomorphism. By the Inverse Function Theorem, there exists an open neighborhood V' of p in M, $V' \subset V$, and $\delta \in (0, \epsilon)$, such that $F|_{U_{(V',\delta)}}$ is a diffeomorphism onto its image $W' := F(U_{(V',\delta)})$, which is an open neighborhood of (p, p) in $M \times M$. There is an open neighborhood W of p in M such that

$$W \times W \subset W' = \bigcup_{q \in V'} \{q\} \times B_\delta(q).$$

Therefore $W \subset B_\delta(q)$ for all $q \in W$. □

Corollary 21.2. *For any $q_1, q_2 \in W$, there is a unique geodesic γ joining q_1 and q_2.*

Corollary 21.3. *Let $\gamma : [a, b] \to M$ be a piecewise smooth curve and write $\gamma(a) = p$ and $\gamma(b) = q$. Suppose that for any piecewise smooth curve $\beta : [c, d] \to M$ such that $\beta(c) = p$ and $\beta(d) = q$, the length of β is at least the length of γ. Then γ is a geodesic.*

Definition 21.4. Let (M, g) be a Riemannian manifold. We say that an open subset $S \subset M$ is *strongly convex* if for each pair q_1, q_2 in the closure $\overline{S}$ of S, there is a unique minimizing geodesic γ such that $\gamma(0) = q_1$, $\gamma(1) = q_2$, and $\gamma((0, 1)) \subset S$.

Example 21.5. Let $(M, g) = (\mathbb{R}^n, g_0)$ be the Euclidean space. Then strongly convex implies convex in the usual sense: $S \subset \mathbb{R}^n$ is convex if for any $q_1, q_2 \in S$, the line segment $\overline{q_1q_2}$ connecting q_1 and q_2 is contained in S. An open ball in

$(\mathbb{R}^n, g_0)$ is strongly convex, thus convex. The set $(0,1)^n$ is convex but not strongly convex.

Proposition 21.6. *For each $p \in M$ there is a $\beta > 0$ such that $B_\beta(p)$ is strongly convex.*

Proof. See [dC, Chapter 3, Section 4]. □

Example 21.7. Let p be any point in the Euclidean space $(\mathbb{R}^n, g_0)$. Then the geodesic ball $B_r(p)$ is strongly convex for $r > 0$.

Let p be a point in the round sphere (S^n, g_{can}) of radius 1. Then the geodesic ball $B_r(p)$ is strongly convex when $0 < r < \pi/2$, but not strongly convex when $\pi/2 \le r < \pi$.

Curvature

Let (M, g) be a Riemannian manifold with ∇ the Levi-Civita connection. Let $\mathfrak{X}(M)$ be the space of smooth vector fields on M.

Definition 21.8. For $X, Y \in \mathfrak{X}(M)$, define an $\mathbb{R}$-linear map $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by the rule

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z = [\nabla_Y, \nabla_X] Z - \nabla_{[Y,X]} Z$$

Proposition 21.9. *The map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $(X,Y,Z) \mapsto R(X,Y)Z$*

(i) *is anti-symmetric in X, Y*
(ii) *is $C^\infty(M)$-linear in X, Y, Z.*

Therefore R can be viewed as an element of

$$\Omega^2(M, \mathrm{End}TM) := C^\infty(M, \Lambda^2 T^*M \otimes T^*M \otimes TM),$$

that is, R is an $\mathrm{End}(TM)$ *valued 2-form on M. In particular, R is a $(1,3)$-tensor.*

Proof. (i) is clear from the definition. Given (i), it remains to show that for any $X, Y, Z \in \mathfrak{X}(M)$ and any $f \in C^\infty(M)$,

(a) $R(fX,Y)Z = fR(X,Y)Z$, and
(b) $R(X,Y)(fZ) = fR(X,Y)Z$

$$\begin{aligned} R(fX,Y)Z &= \nabla_Y \nabla_{fX} Z - \nabla_{fX} \nabla_Y Z + \nabla_{[fX,Y]} Z \\ &= \nabla_Y (f \nabla_X Z) - f \nabla_X \nabla_Y Z + \nabla_{f[X,Y] - Y(f)X} Z \\ &= Y(f) \nabla_X Z + f \nabla_Y \nabla_X Z - f \nabla_X \nabla_Y Z + f \nabla_{[X,Y]} Z - Y(f) \nabla_X Z \\ &= fR(X,Y)Z \end{aligned}$$

$$\begin{aligned} R(X,Y)(fZ) &= \nabla_Y \nabla_X (fZ) - \nabla_X \nabla_Y (fZ) + \nabla_{[X,Y]} (fZ) \\ &= \nabla_Y (X(f)Z + f \nabla_X Z) - \nabla_X (Y(f)Z + f \nabla_Y Z) + ([X,Y]f)Z + f \nabla_{[X,Y]} Z \\ &= \quad YX(f)Z + X(f) \nabla_Y Z + Y(f) \nabla_X Z + f \nabla_X \nabla_Y Z \\ &\quad -XY(f)Z - Y(f) \nabla_X Z - X(f) \nabla_Y Z - f \nabla_Y \nabla_X Z \\ &\quad +(XY(f) - YX(f))Z + f \nabla_{[X,Y]} Z \\ &= fR(X,Y)Z \end{aligned}$$

□

Proposition 21.10 (Bianchi identity)**.** *We have*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

Proof. See [dC] page 91. □

Definition 21.11. For $X, Y, Z, T \in \mathfrak{X}(M)$, define

$$R(X,Y,Z,T) := \langle R(X,Y)Z, T\rangle.$$

Then $R(X,Y,Z,T)$ is $C^\infty(M)$-linear in each slot, so it is a $(0,4)$ tensor.

Proposition 21.12. *The* $(0,4)$ *tensor* $R(X,Y,Z,T)$ *satisfies the following properties.*

(a) $R(X,Y,Z,T) + R(Y,Z,X,T) + R(Z,X,Y,T) = 0$. *(the Bianchi identity)*
(b) $R \in C^\infty\big(M, \mathrm{Sym}^2(\Lambda^2 T^*M)\big)$, *i.e.*
  (b1) $R(X,Y,Z,T) = -R(Y,X,Z,T)$
  (b2) $R(X,Y,Z,T) = -R(X,Y,T,Z)$
  (b3) $R(X,Y,Z,T) = R(Z,T,X,Y)$

Proof. See [dC] page 91-92. □

22. Monday, November 30, 2015

The Riemannian curvature tensor in local coordinates

Let (U, ϕ) be a C^∞ chart in M. Let $(x_1, \ldots, x_n)$ be local coordinates on U. Let T be any (r,s) tensor on M. Then on U,

$$T = \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_s}} T^{i_1\cdots i_r}_{j_1\cdots j_s} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}$$

where $T^{i_1\cdots i_r}_{j_1\cdots j_s} \in C^\infty(U)$.

As a $(1,3)$ tensor,

$$R = \sum_{i,j,k,m} {R_{ijk}}^m dx_i \otimes dx_j \otimes dx_k \otimes \frac{\partial}{\partial x_m},$$

where ${R_{ijk}}^m \in C^\infty(U)$ is determined by

$$R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k} = \sum_l {R_{ijk}}^m \frac{\partial}{\partial x_m}.$$

As a $(0,4)$ tensor,

$$R = \sum_{i,j,k,l} R_{ijkl} dx_i \otimes dx_j \otimes dx_k \otimes dx_l,$$

where

$$R_{ijkl} = R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}) = \langle R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\rangle = \sum_m {R_{ijk}}^m g_{ml} \in C^\infty(U).$$

By Proposition 21.12,

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0, \quad R_{ijkl} = -R_{jikl}, \quad R_{ijkl} = -R_{ijlk}, \quad R_{ijkl} = R_{klij}.$$

We now express ${R_{ijk}}^m$ in terms of the Christoffel symbol Γ^k_{ij}.

$$R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k} = \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} + \nabla_{[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}]} \frac{\partial}{\partial x_k}$$

where $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$, and

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k} &= \nabla_{\frac{\partial}{\partial x_j}} (\sum_l \Gamma_{ik}^l \frac{\partial}{\partial x_l}) \\ &= \sum_l \frac{\partial \Gamma_{ik}^l}{\partial x_j} \frac{\partial}{\partial x_l} + \sum_l \Gamma_{ik}^l \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_l} \\ &= \sum_m \frac{\partial \Gamma_{ik}^m}{\partial x_j} \frac{\partial}{\partial x_m} + \sum_{l,m} \Gamma_{ik}^l \Gamma_{jl}^m \frac{\partial}{\partial x_m} \end{aligned}$$

So

$$R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \frac{\partial}{\partial x_k} = \sum_m (\frac{\partial \Gamma_{ik}^m}{\partial x_j} - \frac{\partial \Gamma_{jk}^m}{\partial x_i} + \sum_l \Gamma_{ik}^l \Gamma_{jl}^m - \sum_l \Gamma_{jk}^l \Gamma_{il}^m) \frac{\partial}{\partial x_m}.$$

$$\boxed{R_{ijk}{}^m = \frac{\partial \Gamma_{ik}^m}{\partial x_j} - \frac{\partial \Gamma_{jk}^m}{\partial x_i} + \sum_l \Gamma_{ik}^l \Gamma_{jl}^m - \sum_l \Gamma_{jk}^l \Gamma_{il}^m}$$

Sectional Curvature

If we fix a point p in a Riemannian manifold (M, g), then $V = T_pM$ is an inner product space.

In general, an inner product on a vector space $V \cong \mathbb{R}^n$ induces an inner product on $\Lambda^2 V$ as follows: if $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V then $\{e_i \wedge e_j : 1 \leq i < j \leq n\}$ is an orthonormal basis of $\Lambda^2 V$. Equivalently, if $x, y \in V$ then

$$|x \wedge y|^2 = \langle x, x \rangle \langle y, y \rangle - \langle x, y \rangle^2.$$

Definition 22.1. Let (M, g) be a Riemannian manifold with p a point of M and σ a 2 dimensional subspace of T_pM. Define the *sectional curvature* of σ, denoted $K(\sigma, p)$, to be

$$K(\sigma, p) = \frac{R(p)(x, y, x, y)}{|x \wedge y|^2}$$

where $\{x, y\}$ is a basis of σ.

This is well-defined because if $\{x', y'\}$ is another basis of σ then $x' = ax + by$ and $y' = cx + dy$ for some

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R}).$$

The by (b1) and (b2) of Proposition 21.12,

$$R(p)(x', y', x', y') = (ad - bc)^2 R(p)(x, y, x, y).$$

We also have

$$x' \wedge y' = (ad - bc) x \wedge y \Rightarrow |x' \wedge y'|^2 = (ad - bc)^2 |x \wedge y|^2.$$

Lemma 22.2. *Let V be an inner product space. Suppose that $r, r' : V \times V \times V \times V \to \mathbb{R}$ are $\mathbb{R}$-linear in each factor and satisfy*

(a) $r(x, y, z, t) + r(y, z, x, t) + r(z, x, y, t) = 0.$
(b1) $r(x, y, z, t) = -r(y, x, z, t).$
(b2) $r(x, y, z, t) = -r(x, y, t, z).$
(b3) $r(x, y, z, t) = r(z, t, x, y).$

Define $K, K' : Gr(2,V) \to \mathbb{R}$ by

$$K(\sigma) = \frac{r(x,y,x,y)}{|x \wedge y|^2}, \qquad K'(\sigma) = \frac{r'(x,y,x,y)}{|x \wedge y|^2}$$

where $\{x,y\}$ is any basis of the 2-dimensional subspace σ of V; this is well-defined by (b1) and (b2). If $K = K'$, then $r = r'$.

Proof. Let $\Delta = r - r' : V \times V \times V \times V \to \mathbb{R}$. Then

(1) Δ is $\mathbb{R}$-linear in each factor.
(2) Δ satisfies (a), (b1), (b2), (b3).
(3) $\Delta(x,y,x,y) = 0$ for any $x, y \in V$.

We want to show that $\Delta \equiv 0$.

For each $x, y, z \in V$, by (3), we have

$$\begin{aligned} 0 &= \Delta(x+z,y,x+z,y) - \Delta(x,y,x,y) - \Delta(z,y,z,y) \\ &= \Delta(x,y,z,y) + \Delta(z,y,x,y) && \text{by linearity} \\ &= 2\Delta(x,y,z,y) && \text{by (b3).} \end{aligned}$$

For any $x, y, z, t \in V$, we have

$$\begin{aligned} 0 &= \Delta(x,y+t,z,y+t) - \Delta(x,y,z,y) - \Delta(x,t,z,t) && \text{by last paragraph} \\ &= \Delta(x,y,z,t) + \Delta(x,t,z,y) && \text{linearity} \\ &= \Delta(x,y,z,t) + \Delta(z,y,x,t) && \text{(b3)} \\ &= \Delta(x,y,z,t) - \Delta(y,z,x,t) && \text{(b1).} \end{aligned}$$

Therefore,

$$\Delta(x,y,z,t) = \Delta(y,z,x,t) = \Delta(z,x,y,t).$$

By (a),

$$\Delta(x,y,z,t) + \Delta(y,z,x,t) + \Delta(z,x,y,t) = 0.$$

We conclude that

$$\Delta(x,y,z,t) = 0$$

for all $x, y, z, t \in V$. This completes the proof. □

Corollary 22.3. *The sectional curvature determines the Riemannian curvature tensor.*

Definition 22.4. We say that (M,g) has *constant sectional curvature* K_0 if for each $p \in M$ and for any $\sigma \in \mathrm{Gr}(2, T_pM)$, we have $K(\sigma) = K_0$.

Lemma 22.5. *Define $r' : V \times V \times V \times V \to \mathbb{R}$ by*

$$r'(x,y,z,t) = \langle x,z\rangle\langle y,t\rangle - \langle x,t\rangle\langle y,z\rangle.$$

Then

(1) *r' is $\mathbb{R}$-linear in each factor*
(2) *r' satisfies (a), (b1), (b2), (b3) in Lemma 22.2.*
(3) *For any $x, y \in V$, we have $r'(x,y,x,y) = |x \wedge y|^2$.*

Corollary 22.6. *The Riemannian manifold (M,g) has constant sectional curvature K_0 if and only if for each $X, Y, Z, T \in \mathfrak{X}(M)$, we have*

$$R(X,Y,Z,T) = K_0\left(\langle X,Z\rangle\langle Y,T\rangle - \langle X,T\rangle\langle Y,Z\rangle\right)$$

Definition 22.7. We say a Riemannian manifold (M, g) is *flat* if its Riemannian curvature tensor is identically zero.

Remark 22.8. By Corollary 22.6, (M, g) is flat if and only if M has constant sectional curvature equal to zero.

Example 22.9. Euclidean space $(\mathbb{R}^n, g_0 = dx_1^2 + \cdots + dx_n^2)$ is flat, since the Christoffel symbols are zero and hence R_{ijkl} are zero. Hence $(\mathbb{R}^n, g_0)$ has constant sectional curvature equal to zero.

Lemma 22.10. *Let $f : (M_1, g_1) \to (M_2, g_2)$ be a local isometry, that is, f is a local diffeomorphism and $f^*g_2 = g_1$. Let R_1 be the curvature tensor of (M_1, g_1) and let R_2 be the curvature tensor of (M_2, g_2). Then $R_1 = f^*R_2$.*

Proof. In terms of local coordinates, we see that the local coordinates are equal and the g_{ij} are equal, hence so are the curvature tensors. □

Example 22.11 (Flat n-torus)**.** There is a local isometry from $(\mathbb{R}^n, g_0)$ to $(T^n = (S^1)^n, g := (g_{\text{can}})^n)$. Therefore (T^n, g) is flat.

Example 22.12.

- At a future time, we will see that (S^n, g_{can}) has constant sectional curvature equal to $+1$. As a consequence, $(S^n, r^2 g_{can})$ (the round sphere of radius $r > 0$) has constant sectional curvature equal to $K = 1/r^2$.
- We will also see that $\mathcal{H}^n = \{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_n > 0\}$ (upper half space) equipped with
$$g_n = \frac{dy_1^2 + \cdots + dy_n^2}{y_n^2}$$
has constant sectional curvature $K = -1$.

Two-dimensional case

Let (M, g) be a 2-dimensional Riemannian manifold. Let (U, ϕ) be a C^∞ chart on M, and let (x_1, x_2) be local coordinates on U. Then on U we have

$$g = g_{11}dx_1^2 + g_{12}dx_1dx_2 + g_{21}dx_2dx_1 + g_{22}dx_2^2 = g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2^2.$$

$$R = \sum_{i,j,k,l=1}^{2} R_{ijkl}dx_i \otimes dx_j \otimes dx_k \otimes dx_l = R_{1212}(dx_1 \wedge dx_2) \otimes (dx_1 \wedge dx_2).$$

The only 2-dimensional subspace of T_pM is itself. So in this case the sectional curvature K is a smooth function on M: $K(p) = K(p, T_pM)$ for $p \in M$.

$$\boxed{K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2}}$$

Example 22.13. $(M, g) = (S^2, g_{\text{can}} = d\phi^2 + \sin^2\phi d\theta^2)$. By Example 17.5,

$$\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\phi} = 0, \quad \nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\theta} = \nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\phi} = \cot\theta\frac{\partial}{\partial\theta}, \quad \nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta} = -\sin\phi\cos\phi\frac{\partial}{\partial\phi}.$$

Let $(x_1, x_2) = (\phi, \theta)$. Then

$$R_{1212} = \langle R(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta}\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta}\rangle$$

where

$$\begin{aligned} R(\frac{\partial}{\partial\phi},\frac{\partial}{\partial\theta})\frac{\partial}{\partial\phi} &= \nabla_{\frac{\partial}{\partial\theta}}\nabla_{\frac{\partial}{\partial\phi}}\frac{\partial}{\partial\phi} - \nabla_{\frac{\partial}{\partial\phi}}\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\phi} + \nabla_{[\frac{\partial}{\partial\phi},\frac{\partial}{\partial\theta}]}\frac{\partial}{\partial\phi} \\ &= 0 - \nabla_{\frac{\partial}{\partial\phi}}(\cot\phi\frac{\partial}{\partial\theta}) + 0 = \csc^2\phi\frac{\partial}{\partial\theta} - \cot^2\phi\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\theta}. \end{aligned}$$

$$R_{1212} = \langle\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\rangle = \sin^2\phi$$

$$g_{11}g_{22} - g_{12}^2 = \sin^2\phi.$$

So

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = 1.$$

Ricci curvature

Definition 22.14. For any $p \in M$, define a symmetric bilinear form Q_p on T_pM by

$$\begin{aligned} Q_p(x,y) &:= \mathrm{Trace}(T_pM \ni v \mapsto R(x,v,y) \in T_pM) \\ &= \sum_{i=1}^{n} R(x,e_i,y,e_i) \end{aligned}$$

for an orthonormal basis $\{e_i\}$ of T_pM. We then define

$$\mathrm{Ric}_p = \frac{1}{n-1}Q_p$$

which is a symmetric $(0,2)$-tensor on (M,g). (Note that this is the same type of tensor as g.)

Why do we use $\frac{1}{n-1}$? Suppose that (M,g) has constant sectional curvature K_0. Then

$$Q_p(x,y) = \sum_{i=1}^{n} K_0\left(\langle x,y\rangle\langle e_i,e_i\rangle - \langle x,e_i\rangle\langle y,e_i\rangle\right) = K_0(n\langle x,y\rangle - \langle x,y\rangle) = (n-1)K_0\langle x,y\rangle.$$

So then $\mathrm{Ric}_p(x,y) = K_0\langle x,y\rangle$.

In terms of local coordinates, we let

$$R(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_k})\frac{\partial}{\partial x_j} = \sum_l R_{ikj}{}^l\frac{\partial}{\partial x_l}.$$

We let

$$R_{ij} := Q(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}) = \mathrm{Trace}(\frac{\partial}{\partial x_k} \mapsto R(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_k})\frac{\partial}{\partial x_j}) = \sum_k R_{ikj}{}^k = \sum_{k,l} R_{ikjl}g^{kl}.$$

Then $Q = \sum_{i,j} R_{ij}dx_i \otimes dx_j$, where $R_{ij} = R_{ji}$. So

$$\mathrm{Ric} = \frac{1}{n-1}\sum_{i,j} R_{ij}dx_i \otimes dx_j, \quad \text{where } R_{ij} = \sum_{k,l} R_{ikjl}g^{kl}.$$

23. Wednesday, December 2, 2015

Scalar curvature

Definition 23.1. Let (M,g) be a Riemannian manifold. The scalar curvature S of (M,g) is a smooth function on M defined as follows. For each point $p \in M$, define a linear map $K_p : T_pM \to T_pM$ by

$$\langle K_p(x), y\rangle = Q_p(x,y).$$

Then K_p is self-adjoint, meaning $\langle K_p(x), y\rangle = \langle x, K_p(y)\rangle$. We then define

$$\begin{aligned} S(p) &:= \frac{1}{n(n-1)}\mathrm{Trace}(K_p) = \frac{1}{n(n-1)}\sum_{i=1}^n Q_p(e_i,e_i) \\ &= \frac{1}{n(n-1)}\sum_{i,j} R(p)(e_i,e_j,e_i,e_j) = \frac{1}{n}\sum_{i=1}^n \mathrm{Ric}_p(e_i,e_i). \end{aligned}$$

where $\{e_1,\dots,e_n\}$ is any orthonormal basis of T_pM.

We see that if (M,g) has constant sectional curvature K_0, we have $\mathrm{Ric} = K_0 g$ and hence $S(p) = K_0$ for all $p \in M$.

In terms of local coordinates, we have

$$n(n-1)S = R^i_i = R_{ij}g^{ij} = R_{ijkl}g^{ik}g^{jl}$$

In the special case, when $n = 2$, we have

$$R = R_{1212}(dx_1 \wedge dx_2) \otimes (dx_1 \wedge dx_2)$$

and

$$\begin{aligned} S &= \frac{1}{2}(R_{1212}g^{11}g^{22} + R_{2112}g^{21}g^{12} + R_{1221}g^{12}g^{21} + R_{2121}g^{22}g^{11}) \\ &= \frac{1}{2}R_{1212}(2g^{11}g^{22} - 2(g^{12})^2) = R_{1212}(g^{22}g^{11} - (g^{12})^2) = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = K. \end{aligned}$$

Covariant derivatives for tensors

References: [dC, Chapter 4 Section 5], [GHL, 2B.3])

Proposition 23.2. *Let ∇ be an affine connection on a smooth manifold M. Let X be a smooth vector field on M and let $\nabla_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ denote the covariant derivative along X. Then ∇_X has a unique extension*

$$\nabla_X : C^\infty(M, T^r_s M) \to C^\infty(M, T^r_s M)$$

such that

(i) $\nabla_X(c(S)) = c(\nabla_X(S))$ *for any tensor S and any contraction c*

(ii) $\nabla_X(S \otimes T) = (\nabla_X S) \otimes T + S \otimes \nabla_X T$ *for any tensors S, T.*

Proof. For $f \in C^\infty(M)$, we must define $\nabla_X f = X(f)$ by the Leibniz rule and (ii). For a $(0,1)$-tensor $\alpha \in \Omega^1(M)$ and a vector field Y, we must have

$$\begin{aligned} X(\alpha(Y)) &= \nabla_X(\alpha(Y)) = \nabla_X(c(Y \otimes \alpha)) = c(\nabla_X(Y \otimes \alpha)) \\ &= c(\nabla_X Y \otimes \alpha + Y \otimes \nabla_X \alpha) = \alpha(\nabla_X Y) + (\nabla_X \alpha)(Y). \end{aligned}$$

This implies that

$$(\nabla_X \alpha)(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y).$$

By (ii) the covariant derivative ∇_X along X on (r,s) tensors is uniquely determined by the covariant derivative on $(1,0)$ tensors (vector fields) and $(0,1)$ tensors (1-forms). In particular, if T is a $(0,s)$-tensor and $Y_1,\dots,Y_s \in \mathfrak{X}(M)$ then

$$\nabla_X T(Y_1,\dots,Y_s) = X(T(Y_1,\dots,Y_s)) - \sum_{i=1}^{s} T(Y_1,\dots,Y_{i-1},\nabla_X Y_i, Y_{i+1},\dots,Y_s).$$

□

Recall that the Lie derivative behaved similarly. In particular, we had

$$L_X T(Y_1,\dots,Y_s) = X(T(Y_1,\dots,Y_s)) - \sum_{i=1}^{s} T(Y_1,\dots,L_X Y_i,\dots,Y_s).$$

This definition does not depend on the connection. However, the definition of $\nabla_X T$ does.

Remark 23.3. Geometrically, the Lie derivative L_X is the derivative of the pullback of a tensor under a flow ϕ_t of a vector field X. Also, there is a geometric interpretation of ∇_X. We take an integral curve γ of X and we look at $\frac{D}{dt}T(\gamma(t))|_{t=0}$.

The map $X \mapsto \nabla_X T$ is $C^\infty(M)$-linear in X, but the map $X \mapsto L_X T$ is $\mathbb{R}$-linear but not $C^\infty(M)$-linear in X.

We may view ∇ as a map

$$\nabla : C^\infty(M, T^r_s M) \to C^\infty(M, T^r_{s+1} M)$$

by the map $T \mapsto \nabla T$ where

$$\nabla T(X_1,\dots,X_{s+1}) = (\nabla_{X_{s+1}} T)(X_1,\dots,X_s).$$

On a coordinate neighborhood U, let $\Gamma^k_{ij} \in C^\infty(U)$ be defined by

$$\boxed{\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma^k_{ij} \frac{\partial}{\partial x_k}}$$

(The right hand side is a sum over k. We will continue to use this summation convention.)

$$(\nabla_{\frac{\partial}{\partial x_i}} dx_j)(\frac{\partial}{\partial x_k}) = \frac{\partial}{\partial x_i}\left(dx_j(\frac{\partial}{\partial x_k})\right) - dx^j\left(\Gamma^l_{ik}\frac{\partial}{\partial x_l}\right) = -\Gamma^j_{ik}.$$

So we find that

$$\boxed{\nabla_{\frac{\partial}{\partial x_i}} dx^j = -\Gamma^j_{ik} dx^k}$$

If T is an (r,s) tensor, then on U we can write

$$T = T^{i_1\dots i_r}_{j_1\cdots j_s} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}.$$

On U we may write

$$\nabla T = (\nabla T)^{i_1\cdots i_r}_{j_1\cdots j_{s+1}} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_{s+1}}.$$

Our goal is to find $(\nabla T)^{i_1\cdots i_r}_{j_1\cdots j_{s+1}}$. We introduce the notation

$$T^{i_1\cdots i_r}_{j_1\cdots j_s, j_{s+1}} = (\nabla T)^{i_1\cdots i_r}_{j_1\cdots j_{s+1}} = (\nabla_{\frac{\partial}{\partial x_{j_{s+1}}}} T)^{i_1\cdots i_r}_{j_1\cdots j_s}.$$

By this notation, we find that

$$\nabla_{\frac{\partial}{\partial x_k}} T = T^{i_1\cdots i_r}_{j_1\cdots j_s,k} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}.$$

On the other hand, we can apply Leibniz rule, and the above boxed equations to find that (see Assignment 12 Problem 4):

$$\boxed{T^{i_1\cdots i_r}_{j_1\cdots j_s,k} = \frac{\partial}{\partial x_k}(T^{i_1\ldots i_r}_{j_1\cdots j_s}) + \sum_{\alpha=1}^{r} \Gamma^{i_\alpha}_{kl} T^{i_1\cdots i_{\alpha-1} l i_{\alpha+1}\cdots i_r}_{j_1\cdots j_s} - \sum_{\beta=1}^{s} \Gamma^{l}_{k i_\beta} T^{i_1\cdots i_r}_{j_1\cdots j_{\beta-1} l i_{\beta+1}\cdots i_s}}$$

Proposition 23.4. *Let ∇ be an affine connection on a Riemannian manifold (M, g). Then ∇ is compatible with g if and only if $\nabla g = 0$.*

Proof. If $\nabla g = 0$, then $\nabla g(X, Y, Z) = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$. But this implies that

$$0 = \nabla g(X, Y, Z) = (\nabla_Z g)(X, Y) = Z(g(X, Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y),$$

which implies that ∇ is compatible with g. This argument is reversible. □

Proposition 23.5. *Let ∇ be an affine connection. Then ∇ is symmetric (that is, $\nabla_X Y - \nabla_Y X = [X, Y]$) if and only if for any 1-form α on M and any vector fields $X, Y \in \mathfrak{X}(M)$, we have*

$$(d\alpha)(X, Y) = (\nabla\alpha)(Y, X) - (\nabla\alpha)(X, Y).$$

Proof. We have

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$

and

$$(\nabla\alpha)(Y, X) = (\nabla_X \alpha)(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y).$$

The claim now follows easily. □

Let ∇ be the Levi-Civita connection on (M, g). For a smooth function f, we get a one-form $\nabla f \in \Omega^1(M)$, defined by

$$(\nabla f)(X) = \nabla_X f = X(f),$$

so

$$\boxed{\nabla f = df.}$$

In particular, we find that

$$\boxed{df = f_{,i} dx^i \qquad f_{,i} = \frac{\partial f}{\partial x_i}}$$

Gradient, Divergence, Hessian, and Laplacian

Definition 23.6. For a smooth function $f \in C^\infty(M)$, we define a vector field $\mathrm{grad}(f) \in \mathfrak{X}(M)$, called the *gradient of* f, by the rule

$$\langle \mathrm{grad}(f), X \rangle = df(X).$$

Write $\text{grad}(f) = \text{grad}(f)^j \frac{\partial}{\partial x_j}$. Then

$$f_{,j} = \frac{\partial f}{\partial x_j} = df(\frac{\partial}{\partial x_j}) = \langle \text{grad}(f), \frac{\partial}{\partial x_j} \rangle = \text{grad}(f)^i g_{ij}$$

Therefore,

$$\boxed{\text{grad} f = f_,{}^i \frac{\partial}{\partial x_i} \qquad f_,{}^i = f_{,j} g^{ij} = \frac{\partial f}{\partial x_j} g^{ij}.}$$

Definition 23.7. For a vector field Y on M, we define a smooth function $\text{div}Y$, called the *divergence of* Y by the rule

$$\text{div}Y = c(\nabla Y)$$

where c denotes contraction.

Write $Y = Y^i \frac{\partial}{\partial x_i}$. Then

$$\nabla Y = Y^i{}_{,j} \frac{\partial}{\partial x_i} \otimes dx_j, \qquad Y^i{}_{,j} = \frac{\partial Y^i}{\partial x_j} + \Gamma^i_{jk} Y^k.$$

Therefore,

$$\boxed{\text{div}Y = Y^i{}_{,i} = \frac{\partial Y^i}{\partial x_i} + \Gamma^i_{ik} Y^k}$$

where $Y = Y^i \frac{\partial}{\partial x_i}$.

Definition 23.8. For a smooth function f, we define a $(0, 2)$-tensor, called the *Hessian of* f by the rule

$$\text{Hess} f = \nabla \nabla f = \nabla df = \nabla(f_{,i} dx^i) = f_{,ij} dx^i \otimes dx^j.$$

We compute that

$$f_{,ij} = \frac{\partial f_{,i}}{\partial x_j} - \Gamma^k_{ji} f_{,k} = \frac{\partial^2 f}{\partial x_j \partial x_i} - \Gamma^k_{ji} \frac{\partial f}{\partial x_k} = f_{,ji}.$$

It follows that $\text{Hess} f$ is a symmetric $(0, 2)$-tensor.

We also compute that

$$\begin{aligned}\text{Hess}(f)(X, Y) &= (\nabla df)(X, Y) = (\nabla_Y df)(X) \\ &= Y(df(X)) - df(\nabla_Y X) = Y(X(f)) - (\nabla_Y X)(f).\end{aligned}$$

Definition 23.9. For a smooth function f, we define a smooth function Δf, called the *Laplacian of* f, by the rule

$$\Delta f = \text{div}(\text{grad} f) = \text{div}(f_,{}^i \frac{\partial}{\partial x_i}) = f_{,i}{}^i = f_{,ij} g^{ij}.$$

Locally the Laplacian is given by

$$\Delta f = g^{ij} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial f}{\partial x_k} \right).$$

In *normal* coordinates at $p \in M$, we know that $g_{ij}(p) = g^{ij}(p) = \delta_{ij}$ and $\Gamma_{ij}^k(p) = 0$. So we can compute that

$$(\mathrm{grad} f)(p) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) \frac{\partial}{\partial x_i}|_p$$

$$(\mathrm{div} Y)(p) = \sum_{i=1}^n \frac{\partial Y^i}{\partial x_i}(p)$$

$$(\mathrm{Hess} f)(p) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(p) dx^i|_p \otimes dx^j|_p$$

$$(\Delta f)(p) = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}(p)$$

24. Monday, December 7, 2015

Curvature of a connection on a vector bundle

Let $E \to M$ be a smooth vector bundle. Recall that a connection ∇ on E is an $\mathbb{R}$-linear map

$$\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$$
$$s \mapsto \nabla s$$

such that for $f \in C^\infty(M)$ and $s \in \Omega^0(M, E)$,

$$\nabla(fs) = df \otimes s + f \nabla s.$$

Given a vector field $X \in \mathfrak{X}(M)$ and a section $s \in \Omega^0(M, E)$, write $\nabla_X s = \nabla s(X) \in \Omega^0(M, E)$. For vector fields $X, Y \in \mathfrak{X}(M)$, define

$$R_\nabla(X, Y) : C^\infty(M, E) \to C^\infty(M, E)$$

by the rule

$$R_\nabla(X, Y)s = \nabla_X \nabla_Y s - \nabla_X \nabla_Y s - \nabla_{[X,Y]} s.$$

Then

(i) $R_\nabla(X, Y) = -R_\nabla(Y, X)$
(ii) $R_\nabla(X, Y)$ is $C^\infty(M)$-linear in X, Y, and s.

We may therefore view R_∇ as an element of

$$\Omega^2(M, \mathrm{End} E) = C^\infty(M, \Lambda^2 T^* M \otimes \mathrm{End} E)$$

We call R_∇ the curvature of ∇.

For a smooth map $f : N \to M$, we get a pullback connection $f^*\nabla$ on the pullback bundle $f^*E \to N$. Then the curvature $R_{f^*\nabla}$ of the pull back connection $f^*\nabla$ is the pull back of the curvature R_∇ of ∇:

$$R_{f^*\nabla} = f^* R_\nabla \in \Omega^2(N, \mathrm{End} f^* E)$$

Jacobi Fields

Let (M,g) be a Riemannian manifold. A Jacobi field $J(t)$ is a smooth vector field along a geodesic $\gamma : I \to M$ which arises in the following way. Consider a smooth map

$$f : (-\epsilon,\epsilon)\times[0,a] \to M$$
$$(s,t) \mapsto f_s(t) = f(s,t)$$

(which we think of as a family of geodesics parametrized by $s \in (-\epsilon,\epsilon)$) such that for any $s \in (-\epsilon,\epsilon)$, the map $f_s : [0,a] \to M$ is a geodesic and such that $f_0 = \gamma$. We then set

$$J(t) = \frac{\partial f}{\partial s}(0,t).$$

Lemma 24.1. *Let $A = (-\epsilon,\epsilon)\times[0,a] \subset \mathbb{R}^2$. Let $f : A \to M$ be any smooth map. Then $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$ are global vector fields on A. Recall that we have defined*

$$\frac{\partial f}{\partial s} := f_*(\frac{\partial}{\partial s}), \quad \frac{\partial f}{\partial t} := f_*(\frac{\partial}{\partial s}) \in C^\infty(A, f^*TM).$$

Let ∇ be the Levi-Civita connection on (M,g) and let $D = f^*\nabla$ be the pullback connection on f^*TM. Then

$$\frac{D}{\partial s}\frac{\partial f}{\partial t} - \frac{D}{\partial t}\frac{\partial f}{\partial s} = 0 \tag{24.1}$$

$$\frac{D^2}{dt^2}\frac{\partial f}{\partial s} - \frac{D}{ds}\left(\frac{D}{dt}\frac{\partial f}{\partial t}\right) + R(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s})\frac{\partial f}{\partial t} = 0 \tag{24.2}$$

Proof. By the symmetric of the pullback connection, we have

$$0 = f_*[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] = D_{\frac{\partial}{\partial s}} f_* \frac{\partial}{\partial t} - D_{\frac{\partial}{\partial t}} f_* \frac{\partial}{\partial s}. \tag{24.3}$$

which can be rewritten as (24.1).

We also have

$$D_{\frac{\partial}{\partial t}} D_{\frac{\partial}{\partial s}} f_* \frac{\partial}{\partial t} - D_{\frac{\partial}{\partial s}} D_{\frac{\partial}{\partial t}} f_* \frac{\partial}{\partial t} + D_{[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}]} f_* \frac{\partial}{\partial t} = f^*R(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})(f_*\frac{\partial}{\partial t}). \tag{24.4}$$

where $[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] = 0$. By (24.3) and (24.4),

$$\frac{D^2}{dt^2}\frac{\partial f}{\partial s} - \frac{D}{ds}\left(\frac{D}{dt}\frac{\partial f}{\partial t}\right) = R(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t})\frac{\partial f}{\partial t},$$

which is equivalent to (24.2). □

We now note that: $f_s : [0,a] \to M$ is a geodesic for any $s \in (-\epsilon,\epsilon)$ if and only if

$$\frac{D}{dt}\frac{\partial f}{\partial t}(s,t) = 0 \quad \text{for any } s,t.$$

Therefore, for a family of geodesics f_s, (24.2) becomes

$$\frac{D^2}{dt^2}\frac{\partial f}{\partial s} + R(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s})\frac{\partial f}{\partial t} = 0$$

In particular, for $s = 0$, if we set

$$\frac{\partial f}{\partial t}(0,t) = \gamma'(t) \quad \text{and} \quad \frac{\partial f}{\partial s}(0,t) = J(t),$$

then we see that

$$\boxed{\frac{D^2 J}{dt^2} + R(\gamma', J)\gamma' = 0.} \tag{24.5}$$

Definition 24.2. A vector field $J(t)$ along a geodesics $\gamma : [0, a] \to M$ is called a *Jacobi field* if it satisfies the Jacobi equation (24.5).

Proposition 24.3. *Let $\gamma : [0, a] \to M$ be a geodesic, with $\gamma(0) = p$ and $\gamma'(0) = v \in T_pM$ (so that $\gamma(t) = \exp_p(tv)$. Then*

(a) *For any $u, w \in T_pM$, there is a unique Jacobi field $J(t)$ along $\gamma(t)$ with $J(0) = u$ and $\frac{DJ}{dt}(0) = w$.*
(b) *If $J(t)$ is a Jacobi field along $\gamma(t)$, then there is a smooth map $f : (-\epsilon, \epsilon) \times [0, a] \to M$ written $f(s, t) = f_s(t)$ such that*
   (i) *for each $s \in (-\epsilon, \epsilon)$, the map $f_s : [0, a] \to M$ is a geodesic,*
   (ii) *$f_0 = \gamma$, and*
   (iii) *$\frac{\partial f}{\partial s}(0, t) = J(t)$.*

Example 24.4. In Proposition 24.3, suppose that $(M, g) = (\mathbb{R}^n, g_0)$ is the Euclidean space, then $\gamma(t) = p + tv$. The Jacobi equation is reduced to $\frac{D^2 J}{dt^2} = 0$. The unique solution in part (a) is given by $J(t) = u + tw$, and the smooth map f in part (b) can be given by $f(s, t) = (p + su) + t(v + sw)$.

Proof of Proposition 24.3.
(a) Let $e_1, \ldots, e_n$ be an orthonormal basis of T_pM and let $e_i(t)$ be parallel transport of e_i along $\gamma(t)$, that is, $e_i(t)$ is the unique parallel vector field along $\gamma(t)$ such that $e_i(0) = e_i$. Then for any $t \in [0, a]$, we see that $\{e_i(t)\}$ is an orthonormal basis of $T_{\gamma(t)}M$. If $J(t)$ is a smooth vector field along $\gamma(t)$, then we may write

$$J(t) = \sum_{i=1}^{n} f_i(t) e_i(t)$$

for some smooth $f_i : [0, a] \to \mathbb{R}$. We see that $J(t)$ is a Jacobi field along $\gamma(t)$ if and only if the Jacobi equation holds, which holds if and only if

$$\sum_{i=1}^{n} f_i''(t) e_i(t) + \sum_{j=1}^{n} f_j(t) R(\gamma'(t), e_j(t))\gamma'(t) = 0.$$

Taking inner product of the above equation and e_i, we see that the above equation is equivalent to

$$f_i''(t) + \sum_{j=1}^{n} f_j(t) R(\gamma'(t), e_j(t), \gamma'(t), e_i(t)) = 0, \quad i = 1, \ldots, n.$$

Define $a_{ij}(t) \in C^\infty([0, a])$ by

$$a_{ij}(t) = R(\gamma'(t), e_j(t), \gamma'(t), e_i(t)).$$

Then $a_{ij}(t) = a_{ij}(t)$. We see that $J(t)$ is a Jacobi field along $\gamma(t)$ if and only if

$$f_i''(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(t) = 0 \quad \text{for } i = 1, \ldots, n$$

if and only if

$$\frac{d^2}{dt^2}\vec{f}(t) + A(t)\vec{f}(t) = 0$$

where $\vec{f}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$, and $A(t)$ is the matrix $(a_{ij}(t))$. We also have

$$\begin{cases} J(0) = u \\ \frac{DJ}{dt}(0) = w \end{cases} \Leftrightarrow \begin{cases} \vec{f}(0) = \vec{u} \\ \frac{d\vec{f}}{dt}(0) = \vec{w} \end{cases}$$

where

$$\vec{u} = \begin{bmatrix} \langle u, e_1 \rangle \\ \vdots \\ \langle u, e_n \rangle \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} \langle w, e_1 \rangle \\ \vdots \\ \langle w, e_n \rangle \end{bmatrix}$$

The uniqueness of ODE's implies there is a unique solution satisfying these conditions.

(b) (cf. [dC] Chapter 5 Exercise 2)

(Idea of the proof: set $u := J(0), w := \frac{DJ}{dt}(0) \in T_pM$. When $(M, g) = (\mathbb{R}^n, g_0)$, we have $f(s,t) = (p + su) + t(v + sw) = \exp_{p+su}(t(v + sw))$. This motivates the construction of $f(s,t)$ in the general case: $f(s,t) = \exp_{\lambda(s)}(t(v(s) + sw(s)))$, where $\lambda(s) = \exp_p(su)$ and $v(s), w(s) \in T_{\lambda(s)}M$ are the parallel tranports of $v, w \in T_pM$ along the curve $\lambda(s)$.)

Let $J(t)$ be a Jacobi field along $\gamma(t) = \exp_p(tv)$. Let $u := J(0), w := \frac{DJ}{dt}(0) \in T_pM$. Define $\lambda : (-\epsilon, \epsilon) \to M$ by $\lambda(s) = \exp_p(su)$. Then $\lambda(0) = 0$ and $\lambda'(0) = u$. Let $v(s)$ (resp. $w(s)$) be the unique parallel vector field along the curve $\lambda(s)$ such that $v(0) = v$ (resp. $w(0) = w$). Define a smooth map $f : (-\epsilon, \epsilon) \times [0, a] \to M$ by

$$f(s,t) = \exp_{\lambda(s)} \big(t(v(s) + sw(s))\big).$$

Then

(i) For any $s \in (-\epsilon, \epsilon)$, $f_s : [0, a] \to M$ defeind by $f_s(t) = f(s,t)$ is the unique geodesic with $f_s(0) = \lambda(s)$ and $f_s'(0) = v(s) + sw(s)$.

(ii) $f_0(t) = \exp_p(tv) = \gamma(t)$.

(iii) $\bar{J}(t) := \frac{\partial f}{\partial s}(0, t)$ is a Jacobi field along $\gamma(t)$.

It remains to show that $\bar{J}(0) = u$ and $\frac{D\bar{J}}{dt}(0) = w$ ($\Rightarrow \bar{J}(t) = J(t)$).

$$f(s, 0) = \lambda(s) \Rightarrow \bar{J}(0) = \frac{\partial f}{\partial s}(0,0) = \lambda'(0) = u.$$

$$\frac{\partial f}{\partial t}(s, 0) = f_s'(0) = v(s) + sw(s) \Rightarrow \frac{D}{\partial s}\frac{\partial f}{\partial t}(s, 0) = w(s).$$

$$\frac{D\bar{J}}{dt}(0,0) = \frac{D}{\partial t}\frac{\partial f}{\partial s}(0,0) = \frac{D}{\partial s}\frac{\partial f}{\partial t}(0,0) = w(0) = w.$$

□

We now consider the special case $u = 0$ in part (b) of the above proof. Say that $J(t)$ is a Jacobi field along $\gamma(t) = \exp_p(tv)$ such that $J(0) = 0$ and $\frac{DJ}{dt}(0) = w$. Applying the construction from part (b) of the proof, we see that $\lambda(s) = p$ (the constant map) and $f(s,t) = \exp_p(t(v + sw))$. We see that

$$\frac{\partial f}{\partial s}(s,t) = (d\exp_p)_{t(v+sw)}(tw)$$

and hence

$$J(t) = (d\exp_p)_{tv}(tw).$$

Proposition 24.5. *Let $\gamma : [0,a] \to M$ be a geodesic with $\gamma(0) = p$ and $\gamma'(0) = v \in T_pM$ (so that $\gamma(t) = \exp_p(tv)$). Let $J(t)$ be a Jacobi field along $\gamma(t)$ such that $J(0) = 0$ and $\frac{DJ}{dt}(0) = w$. Then*

$$J(t) = (d\exp_p)_{tv}(tw)$$

for $t \in [0,a]$.

Lemma 24.6. *Let $\gamma : [0,a] \to M$ be a geodesic and $J(t)$ a Jacobi field along $\gamma(t)$. Then*

$$\langle J(t), \gamma'(t)\rangle = \langle J(0), \gamma'(0)\rangle + t\langle J'(0), \gamma'(0)\rangle$$

where $J'(0) = \frac{DJ}{dt}(0)$.

Proof. Define a smooth function $f : [0,a] \to \mathbb{R}$ by $f(t) = \langle J(t), \gamma'(t)\rangle$. The lemma says $f(t) = f(0) + f'(0)t$. It suffices to show that $f''(t) = 0$.

Recall that because γ is a geodesic, we have $\frac{D}{dt}\gamma'(t) = 0$. Let $J' = \frac{DJ}{dt}$ and $J'' = \frac{D^2J}{dt^2}$. Then

$$\begin{aligned} f' &= \langle J', \gamma'(t)\rangle \\ f'' &= \langle J'', \gamma'\rangle = -\langle R(\gamma', J)\gamma', \gamma'\rangle = R(\gamma', J, \gamma', \gamma') = 0, \end{aligned}$$

where we use the Jacobi equation $J'' + R(\gamma', J)\gamma = 0$. ☐

Remark 24.7. Note that $\gamma'(t)$ and $t\gamma'(t)$ are Jacobi fields along $\gamma(t)$ (by the Jacobi equation). By Lemma 24.6, for any Jacobi field $J(t)$ along $\gamma(t)$, we have

$$J(t) = (\langle J(0), \gamma'(0)\rangle + t\langle J'(0), \gamma'(0)\rangle)\,\frac{\gamma'(t)}{|\gamma'(0)|^2} + J^{\perp}(t)$$

where $J^{\perp}(t)$ is also a Jacobi field along $\gamma(t)$ and

$$\langle J^{\perp}, \gamma'\rangle = 0.$$

25. Wednesday, December 9, 2015

Jacobi fields on a manifold with constant sectional curvature

Let (M,g) be a Riemannian manifold with constant sectional curvature K. Let $\gamma : [0,a] \to M$ be a *normalized* geodesic (i.e. $|\gamma'| = 1$). Let $p = \gamma(0) \in M$ and $v = \gamma'(0) \in T_pM$. Let $J(t)$ be a Jacobi field along $\gamma(t)$ such that

$$J(0) = 0, \quad \frac{DJ}{dt}(0) = w, \quad \langle w, v\rangle = 0.$$

Then $\langle J(t), \gamma'(t)\rangle = 0$ for all $t \in [0,a]$. For any smooth vector field $V(t)$ along $\gamma(t)$,

$$\langle R(\gamma', J)\gamma', V\rangle = K(\langle \gamma', \gamma'\rangle\langle J, V\rangle - \langle \gamma', V\rangle\langle \gamma', J\rangle) = \langle KJ, V\rangle.$$

Therefore $R(\gamma', J)\gamma' = KJ$. So J satisfies

$$\frac{D^2}{dt^2} + KJ = 0.$$

Let $J(t) = f(t)w(t)$, where f is a smooth function on $[0,a]$ and $w(t)$ is the unique parallel vector field along $\gamma(t)$ with $w(0) = w$. Then

$$\frac{D^2J}{dt^2} + KJ = 0, \quad J(0) = 0, \quad \frac{DJ}{dt}(0) = w,$$

are equivalent to

$$f'' + Kf = 0, \quad f(0) = 0, \quad f'(0) = 0.$$

$$f(t)=\begin{cases}\frac{\sin(\sqrt{K}t)}{\sqrt{K}}, & K>0;\\ t, & K=0;\\ \frac{\sinh(\sqrt{-K}t)}{\sqrt{-K}}, & K<0.\end{cases}$$

Therefore, the unique Jacobi field $J(t)$ along $\gamma(t)$ with $J(0)=0$, $\frac{DJ}{dt}(0)=w$, where $\langle w,\gamma'(0)\rangle=0$, is given by

$$J(t)=\begin{cases}\frac{\sin(\sqrt{K}t)}{\sqrt{K}}w(t), & K>0,\\ tw(t), & K=0,\\ \frac{\sinh(\sqrt{-K}t)}{\sqrt{-K}}w(t), & K<0,\end{cases}$$

where $w(t)$ is the unique parallel vector field along $\gamma(t)$ with $w(0)=w$.

Similarly, the unique Jacobi field $J(t)$ along $\gamma(t)$ with $J(0)=u$, $\frac{DJ}{dt}(0)=0$, where $\langle u,\gamma'(0)\rangle=0$, is given by

$$J(t)=\begin{cases}\cos(\sqrt{K}t)u(t), & K>0,\\ u(t), & K=0,\\ \cosh(\sqrt{-K}t)u(t), & K<0,\end{cases}$$

where $u(t)$ is the unique parallel vector field along $\gamma(t)$ with $u(0)=u$.

Taylor Expansion of g_{ij} in local coordinates

Proposition 25.1. *Let (M,g) be a Riemannian manifold and p a point M. Let $\gamma:[0,a]\to M$ be a geodesic with $\gamma(0)=p$ and $\gamma'(0)=v$. (This means that $\gamma(t)=\exp_p(tv)$.) Let $J(t)$ be a Jacobi field along $\gamma(t)$ with $J(0)=0$ and $\frac{DJ}{dt}(0)=w\in T_pM$. (This means that $J(t)=(d\exp_p)_{tv}(tw)$.) Then*

$$\begin{aligned}|J(t)|^2=\langle w,w\rangle t^2-\frac{1}{3}R(v,w,v,w)t^4-\frac{1}{6}(\nabla_v R)(v,w,v,w)t^5\\ +\left[\frac{2}{45}\langle R(v,w)v,R(v,w)v\rangle-\frac{1}{20}(\nabla_v\nabla_v R)(v,w,v,w)\right]t^6+o(t^6).\end{aligned}$$

Corollary 25.2. *If v and w are orthonormal, then*

$$|J(t)|^2=t^2-\frac{1}{3}K(p,\sigma)t^4+o(t^4)$$

where σ is the span of v and w. As a result, we also have (when $t>0$)

$$|J(t)|=t-\frac{1}{6}K(p,\sigma)t^3+o(t^3).$$

We now prove the proposition.

Proof of Proposition 25.1. Let $f=\langle J,J\rangle$. Need to compute $f^{(k)}(0)$ for $0\le k\le 6$.

Note that

$$\begin{aligned} f' &= 2\langle J', J\rangle \\ f'' &= 2\langle J'', J\rangle + 2\langle J', J'\rangle \\ f^{(3)} &= 2\langle J^{(3)}, J\rangle + 6\langle J'', J'\rangle \\ f^{(4)} &= 2\langle J^{(4)}, J\rangle + 8\langle J^{(3)}, J'\rangle + 6\langle J'', J''\rangle \\ f^{(5)} &= 2\langle J^{(5)}, J\rangle + 10\langle J^{(4)}, J'\rangle + 20\langle J^{(3)}, J''\rangle \\ f^{(6)} &= 2\langle J^{(6)}, J\rangle + 12\langle J^{(5)}, J'\rangle + 30\langle J^{(4)}, J''\rangle + 20\langle J^{(3)}, J^{(3)}\rangle. \end{aligned}$$

We now know that $J(0) = 0$ and $J'(0) = w$. We need to compute $J^{(k)}(0)$ for $2 \leq k \leq 5$. But we have the Jacobi equation, so we know that

$$\begin{aligned} J'' &= -R(\gamma', J)\gamma' \Rightarrow J''(0) \\ J^{(3)} &= -R'(\gamma', J)\gamma' - R(\gamma', J')\gamma' \Rightarrow J^{(3)}(0) = -R(v, w)v \\ J^{(4)} &= -R''(\gamma', J)\gamma' - 2R'(\gamma', J')\gamma' - R(\gamma', J'')\gamma' \Rightarrow J^{(4)}(0) = -2(\nabla_v R)(v, w)v \end{aligned}$$

$$J^{(5)} = -R'''(\gamma', J)\gamma' - 3R''(\gamma', J')\gamma' - 3R'(\gamma', J'')\gamma' - R(\gamma', J^(3))\gamma'$$

$$\Rightarrow J^{(5)}(0) = -3(\nabla_v \nabla_v R)(v, w)v + R(v, R(v, w)v)v$$

We then plug these results into the above expressions for $f^{(k)}$ to find

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 0 \\ f''(0) &= 2\langle w, w\rangle \\ f^{(3)}(0) &= 0 \\ f^{(4)}(0) &= -8\langle R(v, w)v, w\rangle \\ f^{(5)}(0) &= -20\langle (\nabla_v R)(v, w)v, w\rangle \\ f^{(6)}(0) &= 12\langle -3(\nabla_v \nabla_v R)(v, w)v + R(v, R(v, w)v)v, w\rangle + 20\langle R(v, w)v, R(v, w)v\rangle \\ &= -36\langle (\nabla_v \nabla_v R)(v, w)v, v\rangle + 32\langle R(v, w)v, R(v, w)v\rangle. \end{aligned}$$

Using the usual Taylor expansion, we find the desired result. □

Proposition 25.1 implies

$$\begin{aligned} &\langle (d\exp_p)_{tv}(u), (d\exp_p)_{tv} w\rangle \\ =&\langle u, w\rangle - \frac{1}{3}R(v, u, v, w)t^2 - \frac{1}{6}(\nabla_v R)(v, u, v, w)t^3 \\ &+ \left[\frac{2}{45}\langle R(v, u)v, R(v, w)v\rangle - \frac{1}{20}(\nabla_v \nabla_v R)(v, u, v, w)\right] t^4 + O(t^5) \end{aligned}$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM. Then

$$\begin{aligned} &\langle (d\exp_p)_v(e_i), (d\exp_p)_v e_j\rangle \\ =&\langle e_i, e_j\rangle - \frac{1}{3}R(v, e_i, v, e_j) - \frac{1}{6}(\nabla_v R)(v, e_i, v, e_j) \\ &+ \left[\frac{2}{45}\langle R(v, e_i)v, R(v, e_j)v\rangle - \frac{1}{20}(\nabla_v \nabla_v R)(v, e_i, v, e_j)\right] + O(|v|^5) \end{aligned}$$

Suppose that $B_\epsilon(p)$ is a geodesic ball with center p and radius $\epsilon > 0$. Then

$$q = \exp_p(\sum_{k=1}^{n} x_k e_k) \in B_\epsilon(q).$$

where $(x_1, \ldots, x_n)$ are the normal coordinates determined by $(e_1, \ldots, e_n)$. Then

$$\frac{\partial}{\partial x_i}\Big|_q = (d\exp_p)_{\sum_{k=1}^n x_k e_k}(e_i).$$

So

$$g_{ij}(x_1, \ldots, x_n) = \langle (d\exp_p)_{\sum_{k=1}^n x_k e_k}(e_i), (d\exp_p)_{\sum_{l=1}^n x_l e_l}(e_j)\rangle$$

On $B_\epsilon(p)$,

$$\nabla R = \sum_{i,j,k,l,m} R_{ijkl,m} dx^i \otimes dx^j \otimes dx^k \otimes dx^l \otimes dx^m$$

and

$$\nabla\nabla R = \sum_{i,j,k,l,m,r,s} R_{ijkl,rs} dx^i \otimes dx^j \otimes dx^k \otimes dx^l \otimes dx^r \otimes dx^s$$

We obtain the following Taylor expansion of g_{ij}:

$$\boxed{\begin{aligned} g_{ij}(x) = \delta_{ij} - \frac{1}{3}\sum_{k,l} R_{ikjl}(p) x_k x_l - \frac{1}{6}\sum_{k,l,m} R_{ijkl,m}(p) x_k x_l x_m \\ - \frac{1}{20}\sum_{k,l,r,s} R_{ikjl,rs}(p) x_k x_l x_r x_s + \frac{2}{45}\sum_{k,l,r,s,m} R_{iklm}(p) R_{jrsm}(p) x_k x_l x_r x_s + O(|x|^5) \end{aligned}}$$

Taylor Expansion of $\sqrt{\det(g_{ij})}$

Let $g(x) = (g_{ij}(x))$. Then

$$g(x) = I + g^{(2)}(x) + g^{(3)}(x) + g^{(4)}(x) + O(|x|^5)$$

where I is the $n \times n$ identity matrix.

$$\sqrt{\det(g(x))} = \exp(\frac{1}{2}\mathrm{Tr}\log(g(x)))$$

where

$$\log(g(x)) = g^{(2)}(x) + g^{(3)}(x) + g^{(4)}(x) - \frac{1}{2} g^{(2)}(x)^2 + O(|x|^5).$$

$$\begin{aligned} -\frac{1}{2}\left(g^{(2)}(x)^2\right)_{ij} &= -\frac{1}{18}\sum_{k,l,r,s,m} R_{ikml}(p) R_{jrms}(p) x_k x_l x_r x_s \\ &= -\frac{1}{18}\sum_{k,l,r,s,m} R_{iklm}(p) R_{jrsm}(p) x_k x_l x_r x_s \end{aligned}$$

$$\begin{aligned} \mathrm{Tr}\log(g(x)) = -\frac{1}{3}\sum_{k,l} R_{kl}(p) x_k x_l - \frac{1}{6}\sum_{k,l,m} R_{kl,m}(p) x_k x_l x_m \\ -\frac{1}{20}\sum_{k,l,r,s} R_{kl,rs}(p) x_k x_l x_r x_s - \frac{1}{90}\sum_{i,k,l,r,s,m} R_{iklm}(p) R_{irsm}(p) x_k x_l x_r x_s + O(|x|^5) \end{aligned}$$

$$\sqrt{\det(g(x))} = 1 - \frac{1}{6}\sum_{k,l} R_{kl}(p)x_k x_l - \frac{1}{12}\sum_{k,l,m} R_{kl,m}(p)x_k x_l x_m$$

$$\sum_{k,l,r,s}\Big(-\frac{1}{40}\sum_{k,l,r,s} R_{kl,rs}(p) - \frac{1}{180}\sum_{i,m} R_{iklm}(p)R_{irsm}(p) + \frac{1}{72}R_{kl}(p)R_{rs}(p)\Big)x_k x_l x_r x_s + O(|x|^5)$$

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Let $S^2 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=1\}$ be the round sphere of radius 1, and let $p=(0,0,1)$ be the north pole. The exponential map $\exp_p : T_pS^2 \to S^2$ sends a circle of radius $\rho>0$ centered at the origin to the circle

$$\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=\sin^2\rho,\ z=\cos\rho\}.$$

Let (ρ,θ) be the polar coordinates on $T_pS^2=\mathbb{R}^2$. Then

$$\exp_p^*(dx^2+dy^2+dz^2) = d\rho^2 + \sin^2\rho d\theta^2.$$

More generally, given $K>0$, let $S^2(\frac{1}{\sqrt{K}}) = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=\frac{1}{K}\}$ be the round sphere of radius $\frac{1}{\sqrt{K}}$, which has constant sectional curvature $K>0$. Let $p=(0,0,\frac{1}{\sqrt{K}})$ be the north pole. The exponential map $\exp_p : T_PS^2(\frac{1}{\sqrt{K}}) \to S^2(\frac{1}{\sqrt{K}})$ sends a circle of radius $\rho>0$ centered at the origin to the circle

$$\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=\frac{\sin^2(\sqrt{K}\rho)}{K},\ z=\frac{\cos(\sqrt{K}\rho)}{\sqrt{K}}\}.$$

Let (ρ,θ) be the polar coordinates on $T_pS^2(\frac{1}{\sqrt{K}})=\mathbb{R}^2$. Then

$$\exp_p^*(dx^2+dy^2+dz^2) = d\rho^2 + \Big(\frac{\sin(\sqrt{K}\rho)}{\sqrt{K}}\Big)^2 d\theta^2.$$

Let (M,g) be a Riemannnian manifold with constant sectional curvature K. Let $\gamma:[0,a]\to M$ be a normalized geodesic, and let $J(t)$ be a Jacobi field along $\gamma(t)$ with $J(0)=0$, $\frac{DJ}{dt}(0)=w$, where $\langle w,\gamma'(0)\rangle$. Then

$$J(t) = f_K(t)w(t),$$

where

$$f_K(t) = \begin{cases} \frac{\sin(\sqrt{K}t)}{\sqrt{K}}, & K>0,\\ t, & K=0,\\ \frac{\sinh(\sqrt{-K}t)}{\sqrt{-K}}, & K<0,\end{cases}$$

Let $B_\delta(p)$ be the geodesic ball with center p and radius $\delta>0$. Define a C^∞ map

$$F:(0,\delta)\times S^{n-1}\to B_\delta(p),\quad (\rho,v)\mapsto \exp_p(\rho v).$$

Then

$$dF_{(\rho,v)} : T_{(\rho,v)}\Big((0,\delta)\times S^{n-1}\Big) = \mathbb{R}\frac{\partial}{\partial\rho}\oplus T_vS^{n-1} \to T_{\exp_p(\rho v)}M$$

is given by

$$\begin{aligned} dF_{(\rho,v)}(\frac{\partial}{\partial\rho}) &= (d\exp_p)_{\rho v}(v)\\ dF_{(\rho,v)}(w) &= (d\exp_p)_{\rho v}(\rho w)\end{aligned}$$

where $w \in T_v S^{n-1} = \{w \in \mathbb{R}^n : \langle v, w\rangle = 0\}$. By Gauss's lemma,

$$\begin{aligned}\langle (d\exp_p)_{\rho v}(v), (d\exp_p)_{\rho v}(v)\rangle &= \langle v, v\rangle = 1,\\ \langle (d\exp_p)_{\rho v}(v), (d\exp_p)_{\rho v}(\rho w)\rangle &= \rho\langle v, w\rangle = 0.\end{aligned}$$

We have

$$(d\exp_p)_{\rho v}(\rho w) = f_K(\rho) w(\rho v)$$

where $w(\rho v) \in T_{\exp_p(\rho v)}M$ is the parallel transport of $w \in T_pM$ along the geodesic $t \mapsto \exp_p(tv)$. So

$$|(d\exp_p)_{\rho v}(\rho w)|^2 = f_K(\rho)^2 |w|^2.$$

Therefore,

$$F^* g = d\rho^2 + f_K(\rho)^2 g_{\mathrm{can}}^{S^{n-1}} = \begin{cases} d\rho^2 + \Big(\dfrac{\sin(\sqrt{K}\rho)}{\sqrt{K}}\Big)^2 g_{\mathrm{can}}^{S^{n-1}}, & K > 0;\\[2ex] d\rho^2 + \rho^2 g_{\mathrm{can}}^{S^{n-1}}, & K = 0;\\[2ex] d\rho^2 + \Big(\dfrac{\sinh(\sqrt{-K}\rho)}{\sqrt{-K}}\Big)^2 g_{\mathrm{can}}^{S^{n-1}}, & K < 0.\end{cases}$$

Conjugate points

See [dC] Chapter 5 Section 3.

Divergence and Laplacian Revisited

Let (M, g) be a Riemannian manifold.

Given a vector field $Y \in \mathfrak{X}(M)$, we may write $Y = Y^i \frac{\partial}{\partial x_i}$ in a coordinate neighborhood U with local coordinates $(x_1, \dots, x_n)$, where $Y^i \in C^\infty(U)$. Then

$$\mathrm{div} Y = Y^i{}_{,i} = \frac{\partial Y^i}{\partial x_i} + \Gamma^i_{ik} Y^k.$$

Lemma 26.1.

$$\mathrm{div} Y = \frac{1}{\sqrt{\det(g)}} \sum_i \frac{\partial}{\partial x_i}(\sqrt{\det(g)} Y^i).$$

Proof.

$$\begin{aligned}\sum_i \Gamma^i_{ik} &= \frac{1}{2}\sum_{i,j} g^{ij}\Big(\frac{\partial}{\partial x_i} g_{kj} + \frac{\partial}{\partial x_k} g_{ji} - \frac{\partial}{\partial x_j} g_{ik}\Big) = \frac{1}{2}\sum_{i,j} g^{ij}\frac{\partial}{\partial x_k} g_{ji}\\ &= \frac{1}{2}\mathrm{Tr}\Big(g^{-1}\frac{\partial}{\partial x_k} g\Big) = \frac{\partial}{\partial x_k}\log\sqrt{\det(g)} = \frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x_k}(\sqrt{\det(g)}).\end{aligned}$$

$$\begin{aligned}\mathrm{div} Y &= Y^i{}_{,i} = \sum_i \frac{\partial Y^i}{\partial x_i} + \sum_k \Big(\frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x_k}(\sqrt{\det(g)})\Big) Y^k\\ &= \frac{1}{\sqrt{\det(g)}}\sum_i \frac{\partial}{\partial x_i}(\sqrt{\det(g)} Y^i).\end{aligned}$$

□

Corollary 26.2. *Let (M, g) be an oriented Riemannian manifold, and let ω be the volume form determined by the Riemannian metric g and the orientation. Then*

$$d(i_Y \omega) = \mathrm{div}(Y)\omega. \tag{26.1}$$

Proof. It suffice to verify this in each coordinate neighborhood U. Choose local coordinates $(x_1, \ldots, x_n)$ compatible with the orientation. Then

$$\begin{aligned} \omega &= \sqrt{\det(g)}dx_1 \wedge \cdots \wedge dx_n, \\ i_Y\omega &= \sum_{i=1}^{n}(-1)^{i-1}Y^i\sqrt{\det(g)}dx_1 \wedge \cdots dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n \end{aligned}$$

$$d(i_Y\omega) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(Y^i\sqrt{\det(g)})dx_1 \wedge \cdots \wedge dx_n \tag{26.2}$$

$$(\mathrm{div}Y)\omega = \mathrm{div}Y\sqrt{\det(g)}dx_1 \ldots dx_n. \tag{26.3}$$

Equation (26.1) follows from (26.2), (26.3), and Lemma 26.1. □

Corollary 26.3. *In local coordinates, the Laplacian of a smooth function f is given by*

$$\Delta f = \frac{1}{\sqrt{\det(g)}}\sum_{i,j}\frac{\partial}{\partial x_i}\Big(\sqrt{\det(g)}g^{ij}\frac{\partial f}{\partial x_j}\Big),$$

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