Topic: Topology and Knots

We would like to explain the notion of a knot to a certain degree of mathematical precision. To do this, we require some ideas from basic point-set topology.

A good (yet extensive) reference for basic point-set topology is the book by Munkres. We won't require too many results from point-set topology though, so a thorough reading of the book is not necessary.

I'll follow closely the first chapter of Cromwell for the (preliminary) definition of a knot.

Contents

1.1 Topological spaces

Definition 1.1. A **topology** on a set X consists of a collection τ of subsets of X satisfying the following three properties.

(i) The subsets X and $\varnothing$ are elements of τ.

(ii) If $\{U_\alpha : \alpha \in A\}$ is a collection of elements of τ, then the union $\cup_{\alpha \in A} U_\alpha$ belongs to τ.

(iii) If $\{U_1, \ldots, U_n\}$ is a finite collection of elements of τ, then the intersection $\cap_{i=1}^n U_i$ belongs to τ.

A **topological space** is a pair (X, τ) where X is a set and τ is a topology on X. For a topological space (X, τ), the elements of τ are called **open subsets of** X.

A topology on a set X should be thought of specifying which points of X are close together. In particular, the points of an open subset U are to be considered close together.

Example 1.2. Let us describe a topology on the finite set $X = \{1, 2, 3\}$. We let

$$\tau = \{\varnothing, \{1, 2, 3\}, \{1\}, \{1, 3\}, \{1, 2\}\}.$$

We claim that τ is a topology on X. The reader can check that axioms (i) through (iii) are satisfied. This is certainly not the only topology on the set X! Indeed the interested reader can count the number of distinct topologies on X. (Answer: 29)

Exercise 1.3. Check that the intersection of any collection of topologies on a space X is a topology on X.

Exercise 1.4. Show by example that the union of two topologies on a space X may not be a topology on X.

Though topologies on a finite set are a good first example to exemplify the axioms of a topology, they are somewhat uninteresting and stray from geometry. We are more interested in topologies on Euclidean space $\mathbb{R}^n$ and its subsets. In fact, $\mathbb{R}^n$ has a standard topology, which we will denote by τ_0, and we will assume that $\mathbb{R}^n$ is equipped with this topology for the remainder of our notes. Let us describe this topology τ_0.

For a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we let

$$|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$$

denote the norm of x. For a positive number $\epsilon > 0$, we let $B_\epsilon(x)$ denote the ball of radius ϵ about x defined by

$$B_\epsilon(x) = \{y \in \mathbb{R}^n : |y - x| < \epsilon\}.$$

We then say that a subset U of $\mathbb{R}^n$ belongs to τ_0 if for each point $x \in U$, there is a positive real number $\epsilon > 0$ such that the ball $B_\epsilon(x)$ is contained in U. We leave it to the reader to verify that τ_0 indeed defines a topology on $\mathbb{R}^n$. We call the elements of τ_0 **open subsets** of $\mathbb{R}^n$. From this point forward, when $\mathbb{R}^n$ is considered as a topological space, we will assume that it is equipped with this standard topology.

Exercise 1.5. The collection τ_0 of subsets of $\mathbb{R}^n$ forms a topology on $\mathbb{R}^n$.

Any subset S of $\mathbb{R}^n$ acquires a topology in a standard fashion, which we describe next. In fact, more generally, any subset S of a topological space (X, τ) inherits a topology τ_S determined by S and τ. This topology is called the **subspace topology**, and it can be described as follows: the elements of τ_S are those subsets of S of the form $S \cap U$ for $U \in \tau$. The reader can check that this indeed defines a topology τ_S on S.

Exercise 1.6. Check that the subspace topology τ_S is indeed a topology on S.

By the preceding remarks, any subset S of $\mathbb{R}^n$ acquires a topology: the open subsets of S are those of the form $U \cap S$ where U is open in $\mathbb{R}^n$. We caution the reader, however, that the open subsets of S can be very different from the open subsets of $\mathbb{R}^n$ itself, as demonstrated in the exercise below.

Exercise 1.7. Let $S = [0, 1] \subset \mathbb{R}$. Equip S with the subspace topology as a subset of $\mathbb{R}$. Show that the set $(1/2, 1]$ is open in S but that $(1/2, 1]$ is not open in $\mathbb{R}$.

There is another topology on $\mathbb{R}^n$, which is called the product topology, and which is a result of realizing that $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ is the Cartesian product of topological spaces. More generally, if $X_1, \ldots, X_n$ are topological spaces, there is a natural topology on $X_1 \times \cdots \times X_n$ called the **product topology**, which is defined as follows: A subset W of $X_1 \times \cdots \times X_n$ is open if it can be expressed as a union

$$W = \bigcup_{\alpha \in A} U_1^\alpha \times \cdots \times U_n^\alpha$$

where each U_i^α is open in X_i. We let the reader check that this indeed defines a topology on $X_1 \times \cdots \times X_n$.

Exercise 1.8. Check that the product topology is indeed a topology.

In this way, we obtain a product topology on $\mathbb{R}^n$. At first glance, this topology seems to determine different open subsets than the standard topology τ_0. However, it turns out that the two topologies coincide. A proof of this is outlined in the following exercise.

Exercise 1.9. This exercise will show that the product topology on $\mathbb{R}^2$ coincides with the standard topology τ_0 on $\mathbb{R}^2$.

(i) A **base** of a set X is a collection $\mathcal{B}$ of subsets of X satisfying the following two properties

(a) The collection $\mathcal{B}$ covers X

(b) For each pair of elements $U, V \in \mathcal{B}$ and each point $x \in U \cap B$, there is another basis element $W \in \mathcal{B}$ such that $x \in W \subset (U \cap V)$.

Verify that a base $\mathcal{B}$ generates a topology $\tau_{\mathcal{B}}$ on X whose elements consist of unions of elements of $\mathcal{B}$. More precisely, a subset U of X belongs to $\tau_{\mathcal{B}}$ if and only if we may write $U = \cup_{\alpha \in A} B_\alpha$ for some collection $\{B_\alpha : \alpha \in A\} \subset \mathcal{B}$.

(ii) Show that the topology $\tau_{\mathcal{B}}$ satisfies the following universal property: If τ is any topology on X such that $\mathcal{B} \subset \tau$, then $\tau_{\mathcal{B}} \subset \tau$. Conclude that $\tau_{\mathcal{B}}$ is the intersection of all topologies on X containing $\mathcal{B}$.

(iii) Check that if $\mathcal{B}$ and $\mathcal{B}'$ are two bases such that $\mathcal{B} \subset \mathcal{B}'$, then $\tau_{\mathcal{B}} \supset \tau_{\mathcal{B}'}$. (Note that the containment is reversing!)

(iv) For $X = \mathbb{R}^2$, check that the collection $\mathcal{B}_0$ of ϵ-balls around points of X form a base for the standard topology τ_0. (Hint: Take an open set in the standard topology and show that it can be written as a union of ϵ-balls.)

(v) For $X = \mathbb{R}^2$, show that the collection $\mathcal{B}'$ of subsets of the form $(a, b) \times (c, d)$ form a basis for the product topology on $\mathbb{R}^2$.

(vi) For $X = \mathbb{R}^2$, verify that $\mathcal{B} \subset \tau_{\mathcal{B}'}$ and that $\mathcal{B}' \subset \tau_{\mathcal{B}}$.

(vii) Conclude that the product topology on $\mathbb{R}^2$ coincides with the standard topology.

1.2 Continuous maps

We now describe maps between topological spaces. These maps should preserve the topological structures in some sense, and the next definition makes more precise in what sense.

Definition 1.10. Let (X, τ) and (Y, σ) be topological spaces. A map $f : X \to Y$ is called **continuous** if for each open subset V of Y, the preimage $f^{-1}(V)$ is an open subset of X.

This definition coincides with the usual definition of continuity for maps between Euclidean spaces. We invite the interested reader to work out the details in the exercise below. It follows that many familiar functions are continuous. .

Exercise 1.11. Prove directly that any constant map $f : X \to Y$ given by $f(x) = c$ for some fixed $c \in Y$ is continuous.

Exercise 1.12. Suppose that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ satisfies the following property

(*) For each point $x \in \mathbb{R}^n$ and each $\epsilon > 0$, there is a $\delta > 0$ so that $f(B_\delta(x)) \subset B_\epsilon(f(x))$.

Show that f is a continuous map of topological spaces. Conclude that functions which are known to be continuous from Calculus are also continuous in this new topological sense.

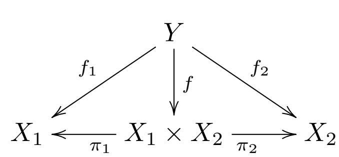
Exercise 1.13. When $X \times Y$ is equipped with the product topology, show that the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous.

Exercise 1.14. Check that the composition of two continuous maps is continuous.

Exercise 1.15. Let X, X_1, X_2 be topological spaces, and for a map $f : X \to X_1 \times X_2$, write $f(x) = (f_1(x), f_2(x))$ for some $f_i : X \to X_i$. Show that f is continuous if and only if both f_1 and f_2 are.

Exercise 1.16. On the other hand, let $f : X_1 \times X_2 \to X$ be a map between topological spaces. For a fixed $x_1 \in X_1$, we may define $f_{x_1} : X_2 \to X$ by $f_{x_1}(x_2) = f(x_1, x_2)$. We may similarly define $f_{x_2} : X_1 \to X$ for a fixed $x_2 \in X_2$. If both f_{x_1} and f_{x_2} are continuous for each $x_1 \in X_1$ and each $x_2 \in X_2$, is it true that f is continuous? Proof or counterexample.

Exercise 1.17. Show that the product topological space $X_1 \times X_2$ satisfies the following universal property: there are continuous maps $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$ such that for each topological space Y and each pair of continuous maps $f_1 : Y \to X_1$ and $f_2 : Y \to X_2$, there is a unique continuous map $f : Y \to X_1 \times X_2$ such that the following diagram commutes



1.3 Homeomorphisms and embeddings

We now describe an equivalence relation on topological spaces, called homeomorphism. This equivalence relation should be considered as a way of saying whether two topological spaces are "topologically the same," meaning that their underlying sets and topological structures on these sets virtually indistinguishable.

A continuous map $f : X \to Y$ is called a **homemorphism** if it admits a continuous inverse $f^{-1} : Y \to X$. In such a case, we say that X is **homeomorphic to** Y. It is routine to check that this defines an equivalence relation. We call the resulting equivalence class of X the **homeomorphism class** of X.

Exercise 1.18. Check that the relation defined by homeomorphism defines an equivalence relation on the set of all topological spaces.

Exercise 1.19. Let (a, b) be an open interval of $\mathbb{R}$. Construct a homeomorphism from (a, b) onto $\mathbb{R}$.

Exercise 1.20. For which real values of c is the map $L(x) = cx$ a homeomorphism from $\mathbb{R}^n$ onto itself?

Exercise 1.21. A space X is called **path-connected** if for any pair of points $x, y \in X$, there is a continuous map $\phi : [0, 1] \to X$ such that $\phi(0) = x$ and $\phi(1) = y$.

(i) Show that the unit circle S^1 is path connected.

(ii) A subset S of $\mathbb{R}^n$ is called **convex** if for any pair of points $x, y \in S$, the line segment $\{tx + (1 - t)y : t \in [0, 1]\}$ belongs to S. Show that a convex subset is path connected.

(iii) On the other hand, find a path connected subset that is not convex.

(iv) Show that the unit circle S^1 is not homeomorphic to the unit interval $(0, 1)$. (Hint: What happens when you remove a point of S^1 and what happens when you remove a point of $(0, 1)$?)

Exercise 1.22. If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are homeomorphisms, construct a homeomorphism $f : X_1 \times X_2 \to Y_1 \times Y_2$.

A continuous map $f : X \to Y$ is called an **embedding** if f is a homeomorphism onto its image $f(X)$ when $f(X)$ is equipped with the subspace topology in Y.

Exercise 1.23. For a positive integer n, construct two embeddings $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}^2$ such that their intersection $\phi_1(\mathbb{R}) \cap \phi_2(\mathbb{R})$ consists of exactly n points.

1.4 Quotient spaces and the unit circle S^1

If X is a topological space, there is a way to form a new topological space from gluing points of X together. For example, if X is the unit interval $X = [0, 1]$, then we might expect that by gluing the endpoints of the interval together, we could form a new topological space, which would look like the unit circle S^1. This process is formulated precisely by the notion of a quotient space.

Let us do the example first, and then the more general construction later. Define an equivalence relation on $X = [0, 1]$ by the rule $x \sim y$ if and only if one of the following three conditions is satisfied

- $x = y$
- $x = 0$ and $y = 1$
- $x = 1$ and $y = 0$.

Let Y denote the set of equivalence classes under this equivalence relation. This means that

$$Y = \{[x] : x \in X\}.$$

Informally, Y consists of those points of X, but we have now identified the points 0 and 1 in Y (and these points only).

The set Y inherits a topology from X in a natural way. In particular, there is a natural projection map $\pi : X \to Y$ that is described by sending an element $x \in X$ to the equivalence class $[x] \in Y$ it represents. A subset V of Y is then declared to be open if and only if its preimage $\pi^{-1}(V)$ is open in X. This can easily be checked to be a topology (see exercise below).

In the general case, if X is a topological space equipped with an equivalence relation $\sim$ and $\pi : X \to Y$ denotes the natural projection to the set Y of equivalence classes, then Y inherits a topology called the **quotient topology**, described by declaring a subset V of Y to be open if and only if its preimage $\pi^{-1}(V)$ is open in X.

Exercise 1.24. Check that the quotient topology is a topology.

Exercise 1.25. Let Y denote the quotient space of $X = [0, 1]$ with the equivalence relation described above, and let $f : [0, 1] \to S^1 \subset \mathbb{C}$ be the map described by $f(t) = e^{2\pi i t}$. Show that f induces a well-defined map $\phi : Y \to S^1$ which is a homeomorphism.

Exercise 1.26. Let X be a topological space with equivalence relation $\sim$, and let $\pi : X \to Y$ denote the natural projection onto the set of equivalence classes Y, equipped with the quotient topology. Show that the quotient map $\pi : X \to Y$ satisfies the following universal property: If Z is another topological space, and $f : Y \to Z$ is any map, then f is continuous if and only if $f \circ \pi$ is continuous.

1.5 Homotopies and isotopies

For fixed topological spaces X and Y, we now describe an equivalence relation on continuous maps from X into Y. This relation makes precise what is meant by saying that a map $f : X \to Y$ can be deformed continuously into another map $g : X \to Y$.

Let $f, g : X \to Y$ be two continuous maps between topological spaces. A **homotopy from f to g** is a continuous map $H : X \times [0,1] \to Y$ satisfying

$$H(x,0) = f(x) \quad \text{and} \quad H(x,1) = g(x) \qquad \text{for } x \in X.$$

For such a homotopy $H : X \times I \to Y$, for a fixed $t \in [0,1]$, we use the notation $H_t : X \to Y$ for $H_t(x) = H(x,t)$. The family of maps H_t should be thought of as a continuous one-parameter family of maps connecting f to g. If there is a homotopy H from f to g, we say that f is **homotopic** to g. The relation determined by homotopy defines an equivalence relation on the set of all continuous map from X into Y. Showing this will require more work than in the case of the relation of homeomorphism, and we let the interested reader work through the details. We let $[f]$ denote the equivalence class of the map $f : X \to Y$ under this relation.

Exercise 1.27. Show that the relation determined by homotopy is an equivalence relation.

Exercise 1.28. Let $B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$ denote the unit ball in $\mathbb{R}^n$. Let $f : B_1(0) \to B_1(0)$ denote the identity map and $g : B_1(0) \to B_1(0)$ by the constant map $g(x) = 0$. Find a homotopy from f to g.

Exercise 1.29. Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$. Let $f : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{R}^{n+1} \setminus 0$ denote the identity map on on the complement of the origin in $\mathbb{R}^{n+1}$. Let $g : \mathbb{R}^{n+1} \setminus 0 \to S^n \subset \mathbb{R}^{n+1} \setminus 0$ denote the map described by

$$g(x) = \frac{x}{|x|}.$$

Show that f is homotopic to g.

Let $f, g : X \to Y$ be embeddings of X into Y. A homotopy $H : X \times [0,1] \to Y$ from f to g is called an **isotopy** if every corresponding $H_t : X \to Y$ defines an embedding of X into Y. This means that an isotopy from f to g is a continuous family of embeddings connecting the embedding f to the embedding g. The relation of isotopy determines an equivalence relation on the set of all embeddings of X into Y. An equivalence class under this relation is called an isotopy class.

1.6 Knots and knot invariants

We are now in a position to define a knot.

Definition 1.30. A knot is a subset of K of $\mathbb{R}^3$ that is homeomorphic to S^1.

For example, the trivial knot is a circle, which can be viewed as the image of the continuous map $t \mapsto (\cos(t), \sin(t), 0)$.

However, this definition is not the full story. We will see later that there is a small subtlety to this definition, involving a property called local flatness, which we will iron out later.

Moreover, we would like to consider knots up to some sort of equivalence. Indeed, just as strings bend and can be moved, we want our knots to be flexible as well. So we need to introduce some sort of equivalence relation that captures this flexibility.

This equivalence relation will involve the notions of homotopy and isotopy. However, we claim that neither homotopy nor isotopy will suffice, so we will require a new notion of ambient isotopy.

To see why neither homotopy nor isotopy suffice, consider a knot $K \subset \mathbb{R}^3$, and let $f : S^1 \to \mathbb{R}^3$ denote the corresponding embedding of S^1. I claim that there is a homotopy from f to the trivial knot $g : S^1 \to \mathbb{R}^3$. Indeed, just take any continuous deformation of the knot to the planar circle in $\mathbb{R}^3$. Moreover, I claim that we can make this homotopy into an isotopy. Indeed, imagine stretching an outer section of the knot K very far, so that K looks almost like the unit circle, except for a small section where all of the knotting happens. Then imaging pulling the knot tighter and tighter until the knotted section shrinks to no thickness at all, thereby obtaining the trivial knot. This procedure describes an isotopy from K to the trivial knot, known as bachelor's unknotting.

Therefore, we require a different type of equivalence, which is called an *ambient isotopy*, namely, an isotopy of the ambient space $\mathbb{R}^3$.

Definition 1.31. Two knots K_1 and K_2 are called **equivalent**, if there is a continuous map $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$, written $H(x,t) = H_t(x)$

- $H_t : \mathbb{R}^3 \to \mathbb{R}^3$ is injective for all $t \in [0,1]$
- $H_0(K_1) = H(K_1, 0) = K_1$ and $H_1(K_2) = H(K_2, 1) = K_2$.

The map H is called an **ambient isotopy**.

Exercise 1.32. Check that ambient isotopy describes an equivalence relation on knots.

A **knot type** is an equivalence class of knots under the equivalence relation of ambient isotopy. We sometimes abuse notation and use the term "knot" to mean an equivalence class of knots. We hope that no confusion will arise.

Definition 1.33. A **knot invariant** with values in S is a function from the set of knots to the set S whose value depends only on knot type.

Clearly, if two knots K_1 and K_2 are equivalent and μ is any knot invariant, then $\mu(K_1) = \mu(K_2)$. However, it may not be true that if $\mu(K_1) = \mu(K_2)$, then K_1 and K_2 are equivalent. Thus, to study differences in knots, we can try to study knot invariants, which is in fact one of our main goals this summer.