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1 Split Links

The first and simplest decomposition that we will look at is that of links of "distant union" as Cromwell refers to them. We can decompose these links by "splitting" them.

Definition 1.1. A link L is **split** if there is a 2-sphere S embedded in $\mathbb{R}^3\backslash\mathrm{L}$ so that there are some components of L on each side of S. If we denote the two components of $\mathbb{R}^3\backslash\mathrm{S}$ by U_1 and U_2 and let $L_i = U_i \cap L$ then we can say $L = L_1 \bigsqcup L_2$ and that L_1 and L_2 are the **split components** of L.

Through this method of splitting we can effectively place the split components of a link in their own topolocial spaces homeomorphic to $\mathbb{R}^3$. By working backwards, we can also create a link as the distant union of L_1 and L_2 by placing these two links into disjoint balls in $\mathbb{R}^3$.

2 Satellite Constructions

We will introduce a lot of terminology in the first part of this section. In order to do this we will also need a few definitions or at least intuitions from parts of chapter 2 that we never looked at. We wil begin with a qiuck look at these.

Definition 2.1. If a loop γ embedded in a surface F does not bound a disc in F it is called **essential**.

Definition 2.2. Let X be a set of disjoint loops embedded in a surface F. If $\lambda \in X$ is a loop which bounds a disc $\triangle \subset F$ such that $\triangle \cap X = \partial\triangle = \lambda$ then λ is called an **innnermost loop**, and $\triangle$ is called an **innermost disc**.

2.1 Satellites and their Companions

Definition 2.3. Let W be a solid torus. A disc properly embedded in W whose boundary is an essential loop in ∂W is called a **meridional disc**. A (possibly knotted) simple loop $\lambda \subset W$ is said to to be **essential** if it meets every meridional disc in W.

Definition 2.4. Let $P \subset W$ be a link embedded in an unknotted solid torus so that at least one component of P is an essential loop in W. Let C be a knot and let V denote a solid tubular neighborhood of C. Choose a homeomorphism $h : W \longrightarrow V$. Then the image $S = h(P)$ is a new link, which is called the **satellite** with **companion** C and **pattern** P. We also note that the solid torus V or its boundary ∂V are often reffered to as the **companion torus** of S.

The process of constructing these satellites is somewhat difficult to understand without visual aids. I will refer you to Cromwell pages 79 and 80 for a couple of pictures. You should imagine using a homeomorphism to form the "tubing" around a knot from a solid torus to help convince yourself that this process even makes sense.

We move on to consider what Cromwell calls a "coordinate system" on the boundary of an embedded solid torus for the purpose of describing nicer satellites. This system consists of meridians and longitudes of the torus that are defined in a way slightly broader than one might expect.

2.2 The Toral Coordinate System

Definition 2.5. Given an embedded solid torus V, we can define a **meridian** to be an essential loop in ∂V that bounds a (meridional) disc in V. A **longitude** is a simple loop in ∂V that meets a meridian once.

Cromwell makes a point of noting that any two merdians are isotopically equivalent, but we can choose one of many distinct longitudes for a coordinate system.

Definition 2.6. The **framing** of a coordinate system (the system set forth by our choice of merdian and longitude) on V is the linking number of the chosen longitude with the core of V. If the framing of a coordinate system is zero, the longitude is sometimes called the **preferred** longitude. If h maps a meridian and preferred longitude of one torus W onto a meridian and preferred longitude of another torus V, it is called a **faithful** homeomorphism.

To move on to the next definition, one must be looking at the left image in Figure 4.1 in Cromwell. We will call this image T.

Definition 2.7. A satellite formed with the pattern from T is called the **double** of companion C. Moreover, if h is faithful it is called an **untwisted double**. If the pattern for a satellite S is a (p,q) torus knot and h is faithful, then S is called a **(p,q) cable** of the companion knot C.

We move on to the only actual result of the section, and it essentially says that this process of satellite construction can never be used to unknot.

2.3 No Unknotting

Theorem 2.8. *The trivial knot has no non-trivial companions.*

We will look at a series of assertions that Cromwell says proves this result. Some of the steps can be difficult to visualize or support.

Proof

1) We start by assuming we a trivial knot K and a companion C, and we call the companion solid torus V and let $T = \partial V$. Since K is trivial, we can assume it is embedded in a 2-sphere which we call S.

2) We now consider $T \cap S$. We can also assume T is isotoped in $\mathbb{R}^3$ so that it meets S transversly (that the two surfaces are not tangent anywhere. The intersection of the two surfaces must be a set of simple loops. This can be seen to be true by considering what the boundary of any intersection of S and T must be. From class we know that boundary must be empty since the neither of the surfaces themselves contain a boundary. This means arcs cannot be in the intersection. It must be a non-empty intersection, however, since $K \subset S$ and K is an essential loop in V.

3) Choose a loop $\lambda \in T \cap S$ which bounds an innermost disc $\triangle$ in S so that $\triangle \cap K = \varnothing$. The fact that we can do this is not trivial. We know from fact 2.5.1 in Cromwell that there are at least two innermost discs in S to choose from. We know K cannot be in both of the discs because the two discs are disjoint, and their boundaries cannot intersect K because K is in the interior of V and their boundaries are in T.

4) Now that we have chosen a λ we will eliminate all elements of $T \cap S$ that bound discs in T. If λ bounds a disc in T consider cutting T along λ and attaching a copy of $\triangle$ to each boundary. The result will be a spere (which we disregard) and a torus (clearly with the same knot structure as our original T) which will call T with a little abuse of notation. Now $T \cap S$ has one less element, and we can repeat this process until we have chosen a λ that does not bound a disc in T.

5) Finally since λ is a simple, essential loop in T, it must be either a merdian or a longitude. It cannot be a merdian, however from the way we have chosen λ because then disc $\triangle$ that it would bound would be a meridional disc of V. Thus K would meet this disc at at least one point, but we chose $\triangle$ to not meet K anywhere, thus we have a contradiction. Since λ must be a longitude of T, it is ambient isotopic to the companion knot C of K. But, λ is also the boundary of a disc in S, which means it must be unknotted. Therefore C is the trivial knot. $\square$

Definition 2.9. A companion of a non-trivial link is a **proper** companion if it is not the trivial knot, and is not equal to any of the link's components.