Factorizing Links & Prime Satellites
4.3 & 4.4

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*The figures on pages 82 and 83 in Cromwell may be useful to accompany many of these definitions.

4.3 Factorizing Links

Definition: Product Link

Let S be a 2-sphere which meets a link, L, transversely at exactly two points. Let $\alpha \subset S$ be any arc in S that connects the points of $L \cap S$. Let U_1 and U_2 be the two components of $\mathbb{R}^3 \setminus S$. Define two new links:

$$L_i = (L \cap U_i) \cup \alpha \quad \text{for } i = 1 \text{ and } 2.$$

L is said to be a **product link** with *factors* L_1 and L_2. S is called a *factorizing sphere.* Symbolically, we write:

$$L = L_1 \# L_2$$

Note: Each factor of a product link is also a companion of it: the companion torus is called a *swallow-follow torus.*

Note: The trivial knot is a factor of every link.

Theorem: The trivial knot has no non-trivial factors.

> **Proof:** The factors of a knot are also its companions. Since we proved last time that the trivial knot has no non-trivial companions, it must likewise have no non-trivial factors. QED.

Definition: Proper Factor

A factor of a link is a **proper factor** if it is not the trivial knot and it is not equal to the link, itself.

- A link with proper factors is called *composite.*
- A link with no proper factors is called *locally trivial.*

Definition: Prime Link

A link is a **prime link** if it is non-trivial, non-split, and locally trivial.

Definition: Simple Link

A non-trivial link is a **simple link** if it is prime and has no proper companions

Note: Any knot with no proper companions is prime. However, for links, this is not the case.

4.4 Prime Satellites

Definition: Wrapping/Winding Number

Let W be an unkotted solid torus containing a pattern link, P. One measure of the complexity of a pattern is the minimum number of intersections P makes with any meridional disc of W. This can be considered either absolutely or algebraically (in which case P is oriented and the sign of the intersection is taken into account). The minimum absolute intersection number is called the **wrapping number** of the pattern. The algebraic intersection number is called the **winding number** of the pattern.

> **Note:** for the algebraic intersection number, we do not need to take a minimum since the algebraic intersection number is the same for all meridional discs in general positions with respect to P.

Classifications by Wrapping Number:

Let ω_P denote the wrapping number of a pattern, P

$\omega_P = 0 \implies$ P sits inside a ball in W and the satellite is unchanged (i.e. $S = P$)

$\omega_P = 1 \implies$ the companion torus is a swallow-follow torus and the satellite construction produces a product link

$\omega_P \geq 2 \implies$ $P \subset W$ is called a *proper pattern*

$\implies$ if a satellite has a proper pattern and the companion knot is non-trivial, then S is called a *proper satellite*

Theorem: A proper satellite is prime if its pattern is a prime knot or the trivial knot.

> **Proof:** Let K be a satellite knot with companion solid torus, V, and pattern, $P \subset W$. Since K is a proper satellite, every meridional disc of V meets K in at least two points. Let S be a factorizing sphere which decomposes K as a product. We assume that K, S, and ∂V are in general position.

We now consider the intersection of two surfaces: the sphere, S, and the torus, ∂V. Since there are both closed surfaces, the intersection must be a (possibly empty) set of loops.

Suppose that $S \cap \partial V \neq \varnothing$ and choose a loop, $\lambda \in S \cap \partial V$, which bounds a disc, Δ in S, s.t. $\Delta \cap \partial V = \partial\Delta = \lambda$. Since there are at least two innermost discs and $K \cap S$ contains only two points, it is possible to choose Δ so that it meets K in at most one point.

If λ bounds a disc in ∂V, then we can perform surgery to remove the intersection (as we did in a proof yesterday). Therefore, we can assume that no loops in $S \cap \partial V$ bound discs in ∂V.

This leaves us with two cases to consider:

1. Suppose Δ is outside V and λ is a longitude of ∂V. Any longitude of ∂V is isotopic to the companion knot which, since it is spanned by the disc, Δ, must be the trivial knot. This contradicts the fact that K is a proper satellite.

2. Suppose Δ is inside V and λ is a meridian of ∂V. Since every meridional disc of V meets K in at least two points, Δ must contain both points of $K \cap S$, contrary to assumption.

Both cases lead to a contradiction so there is no such loop, λ, and $S \cap \partial V = \varnothing$.

Therefore, S lies inside V and its preimage, $h^{-1}(S)$ in W decomposes the pattern, P, as a product. This is also a contradiction and the factorizing sphere cannot exist. Therefore, the pattern must either be prime or trivial.

Corollary: Doubles of non-trivial knots are prime.

Proof: The pattern is the trivial knot.

Corollary: Cable knots are prime.

Proof: Torus knots are prime.