Uniqueness of Factorization & The Product Operation
4.5 & 4.6

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4.5 Uniqueness of Factorization

Big Picture

The big focus of this chapter is to show that the factorization operation that we have defined on knots and links closely resembles the intuition that we get from our factorization operation on positive integers. In particular, we will show that the factorization operation on knots satisfies numerous properties that are similar to those that our satisfied by the factorization operation on positive integers.

Natural Numbers	Knots and Links
1. An integer has a finite number of factors.	1. A knot has a finite number of factors.
2. If a is prime and $a \mid bc$, then $a \mid b$ or $a \mid b$.	2. If a prime knot is a factor of the product of two knots, then it is a factor of one of the two factors.
3. If $a \neq 0$ and $ab = ac$, then $b = c$.	3. If K_P is prime and $K_P \# K_A = K_P \# K_B$, then $K_A = K_B$.
4. Every integer greater than 1 is the unique product of primes.	4. The prime factors of a knot are uniquely determined up to order.

Note: Although the proofs that follow concentrate on knots, the results of these proofs generalize to links.

Definition: Complexity Function

A **complexity function** χ from the set of knots to a discrete ordered set (like $\mathbb{N}_0$) must satisfy the following properties:

1. $\chi(\circ) = 0$
2. $\chi(\circ) < \chi(K)$ whenever K is not the trivial knot
3. $\chi(K_1) < \chi(K_1 \# K_2)$ whenever K_2 is not the trivial knot

In a later chapter, we will define the genus, a link invariant that satisfies this definition of a complexity function.

Theorem 1: A knot has a finite number of factors.

Proof. The complexity function is reduced each time a knot is factorized. Furthermore, it is bounded below by $\chi(\circ)$, 0. This means that the existence of the complexity function implies that the factorization of a given knot is terminated in finitely many steps. ■

Theorem 2: Let K be a knot which factorizes as $K_A \# K_B$. Let K_P be a prime knot which is a factor of K so that K can also be decomposed as $K_P \# K_Q$. Then one of the following holds:

1. K_P is a factor of K_A and K_B is a factor of K_Q : $K_A = K_P \# K_C$ and $K_C \# K_B = K_Q$ for some knot K_C.

2. K_P is a factor of K_B and K_A is a factor of K_Q : $K_B = K_P \# K_C$ and $K_C \# K_A = K_Q$ for some knot K_C.

Proof. Let U be a ball such that ∂U is a sphere which factorizes K as $K_P \# K_Q$, and $K \cap U$ becomes the factor K_P when completed by an arc in ∂U. Let S be the sphere that factorizes K as $K_A \# K_B$. We consider two cases.

1. If $S \cap \partial U = \emptyset$, then K_P must be a factor of either K_A or K_B and the other must be a factor of K_Q. This immediately leads to the result we seek.

2. We now turn our attention to the general case that S intersects ∂U in a set of loops that are disjoint from each other and from the knot K. Since $K \cap S \cap \partial U = \emptyset$, then the four points $(K \cap S) \cup (K \cap \partial U)$ are distinct.

   Suppose $\lambda \in S \cap \partial U$ is a loop that is innermost on S and bounds a disc $\Delta \subset S$ such that $\Delta \cap \partial U = \partial \Delta = \lambda \, \text{and} \, \Delta \cap K = \emptyset$. The loop λ bounds two discs in ∂U, one of which must contain both points of $K \cap \partial U$. Let Δ_U be the other disc so that $\Delta_U \subset \partial U$, $\partial \Delta_U = \partial \Delta$ and $\Delta_U \cap K = \emptyset$.

   $\Delta_U \cup \Delta$ is a sphere that does not meet K. The disc Δ_U can be isotoped in $S^3 - K$ by pushing it across the ball bounded by $\Delta_U \cup \Delta$ until it lies just beyond Δ. This removes the loop λ from the set $S \cap \partial U$.

   This means that we can assume all loops in $S \cap \partial U$ have linking number ± 1 with K. This means that on each factorizing sphere, the loops are arranged in parallel and separate the two points of K. Thus S is divided into two polar discs and some annuli.

   Suppose one of the discs lies inside U. Then there is a disc $\Delta \subset S$ such that $\Delta \subset U$ and $\Delta \cap \partial U = \partial \Delta$. The disc Δ meets K in a single point and divides U into two arc-ball pairs. Since K_P is prime, one of these arc-ball pairs must be trivial and Δ can be isotoped to lie outside U without affectng the decompositions of K into $K_A \# K_B$ and $K_P \# K_Q$.

   Thus, we may assume that every component of $S \cap U$ is an annulus. Let $R \subset S \cap U$ be an annulus which is outermost in U, meaning that it is furthest from K. Then the two loops of ∂R also bound an annulus $R_U \subset \partial U$. Together, these two annuli form a torus $T = R \cup R_U$. If this torus bounds a solid torus in U, then R is parallel to R_U and can be isotoped in $S^3 - K$ to lie outside U, thus reducing the number of intersections of S with ∂U. Otherwise T bounds a ball with a knotted hole in U and is a swallow-follow torus for K which follows the arc $K \cap U$ and swallows the rest of K. In other words, T follows K_P and swallows K_Q.

Let U_A be the ball bounded by S such that $K \cap U_A$ becomes the factor K_A when completed by an arc in S. By a small isotopy in $S^3 - K$, R can be moved slightly so that $R \cup R_U$ does not meet S. Thus the swallow-follow torus lies entirely on one side of S. Suppose that $T \subset U_A$. Then T is also a swallow-follow torus for K_A and th followed factor, K_P must be a factor of K_A. Let K_C denote the factor of K_A that is swalloed by T so that $K_A = K_P \# K_C$. In S^3, the torus swallows K_C and K_B. Hence $K_Q = K_C \# K_B$.

■

Theorem 3: Let K_P be a prime knot and suppose $K_P \# K_Q = K_A \# K_B$. If $K_P = K_A$, then $K_Q = K_B$.

Proof. If we apply Theorem 2, we have two cases to consider.

1. First, suppose that there exists a knot K_C such that $K_A = K_P \# K_C$ and $K_C \# K_B = K_Q$. Using the complexity function from Theorem 1 on the factorization of K_A, we have that

$$\chi(K_P) = \chi(K_A) = \chi(K_P \# K_C) \geq \chi(K_P)$$

   Equality holds only when K_C is the trivial knot. Therefore,

$$K_Q = \circ \# K_B = K_B$$

2. Now suppose that there exists a knot K_C such that $K_B = K_P \# K_C$ and $K_C \# K_A = K_Q$. Since, we assumed that $K_A = K_P$, then K_B and K_Q have the same factors: both are equal to $K_P \# K_C$.
   Therefore, $K_B = K_Q$.

■

Theorem 4: The factors of a knot are uniquely determined up to order.

Proof. Suppose that a knot K is factorized into prime knots in the following two ways:

$$K = A_1 \# ... \# A_m \quad \text{and} \quad K = B_1 \# ... \# B_n$$

We need to show that $n = m$ and that there is a permutation π of the integers $1, ..., n$ such that $A_i = B_{\pi(i)}$.
If K is the trivial knot then $m = n = 0$.
Suppose that $m = 1$. From Theorem 2, A_1 must be a factor of B_1 or of $B_2 \# ... \# B_n$. Repeating this argument inductively, A_1 must be a factor of B_i for some i. Since B_i is prime, $A_1 = B_i$. By Theorem 3, we can essentially cancel A_1 and B_i from the equation. We then have that

$$\circ = B_1 \# ... \# B_{i-1} \# B_{i+1} \# ... \# B_n$$

Since the trivial knot has no non-trivial factors, for all $j \neq i B_j$ must be trivial. Therefore, $m = n = 1$ and $A_1 = B_1$.
We can repeat this argument for the general case and use induction on m to complete the proof. ■

4.6 Product Operation

Big Picture

Factorization has a partial inverse. Factorization allows us to reduce a link L down to its prime factors. We may reverse this process with the product operation. Namely, the product operation allows us to form a link L with given factors L_1 and L_2.

Definition: Product Operation

Let L_1 and L_2 be two links. Let S be a sphere that separates the two constitutents of the split link $L_1 \sqcup L_2$. Choose a rectangular disc R whose boundary is composed of four arcs, $\partial R = a, b, c, d$, such that $L_1 \cap R = a$ and $L_2 \cap R = c$ and $R \cap S$ is a single simple arc, implying that b and d each meet S in a single point.
The **product** or **connected sum** L of L_1 and L_2 is formed by switching the arcs in ∂R. More formally,

$$L = L_1 \# L_2 = (L_1 - a) \cup (L_2 - c) \cup b \cup d$$

Theorem: Given a non-trivial knot K, there is no 'anti-knot' K^{-1} such that the product $K \# K^{-1}$ is the trivial knot.

Proof. This is a corollary to an earlier theorem which states that the trivial knot has no non-trivial factors. The existence of a product as written in the statement above would contradict this result. ■

Theorem: $(\mathbb{K}, \#)$ is an Abelian semigroup with unit and unique factorization.

An **Abelian semigroup** is a set whose elements are related by a binary operation that is closed, associative, and commutative. The set $(\mathbb{K}, \#)$ satisfies all three of these properties.