Genus and Factorization
5.6

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Definition: Compressible

A surface, F, embedded in a 3-manifold, M, is **compressible** if any one of the following is satisfied:

(i) F is a 2-sphere and it bounds a 3-ball in M.

(ii) F is a disc in ∂M.

(iii) F is a disc properly embedded in M and there is a 3-ball in M whose boundary is contained in $F \cup \partial M$.

(iv) F is not a 2-sphere or a disc and there is a disc, $\Delta \subset M$, s.t. $\Delta \cap F = \partial\Delta$ and $\partial\Delta$ is an essential loop in F. The disc, Δ, is called a *compressing disc*.

Hence, a surface is *incompressible* if it does not satisfy any of the above requirements.

Theorem: Suppose that a link, L, has a connected, incompressible, spanning surface of minimal genus. If L can be factorized as $L_1 \# L_2$, then:

$$g(L_1 \# L_2) = g(L_1) + g(L_2)$$

Proof: Let F_i be a minimal genus, spanning surface for L_i. Take a 2-sphere, $S \subset \mathbb{R}^3$, which separates $\mathbb{R}^3$ into two pieces, U_1 and U_2, s.t. $U_1 \cup U_2 = \mathbb{R}^3$ and $U_1 \cap U_2 = \partial U_i = S$ and s.t. $F_i \subset U_i$ with $F_1 \cap S = F_2 \cap S$ equals a single, simple arc. Then $F_1 \cup F_2$ is a spanning surface for $L_1 \# L_2$. Hence:

$$g(L_1) + g(L_2) = g(F_1) + g(F_2) = g(F_1 \cup F_2) \geq g(L_1 \# L_2).$$

To show the reverse inequality, we let F be a connected, incompressible, minimal genus surface spanning $L = L_1 \# L_2$ and let S be a factorizing sphere. Assume that F and S are in general position. The two surfaces intersect in an arc, α, connecting two points of $L \cap S$ and a (possibly empty) set of loops.

Let $\lambda \in F \cap S$ be a loop which is innermost on S and bounds a disc, $\Delta \subset S$, s.t. $\Delta \cap F = \partial\Delta = \lambda$. There must be such a loop since α lies on only one side of any loop in S.

If λ bounds a disc in F, then we can perform surgery to simplify the situation: cut F along λ and attach a copy of Δ to each boundary. The result of the surgery is a sphere (which we discard) and a surface spanning L which we continue to call F. This procedure reduces the number of intersections of F with S and it can be repeated until there are no loops in $F \cap S$ that bound discs in F.

This leaves us with two cases to consider:

(i) If λ is separating in F but does not bound a disc in F, then Δ is a compressing disc for F.

(ii) If λ is a non-separating curve in F, then Δ is, again, a compressing disc.

Both cases contradict the assumption that F is incompressible, so there is no such loop, λ, and $F \cap S = \alpha$.

Let F_1 and F_2 be the two components of $F \backslash \alpha$ which are spanning surfaces for L_1 and L_2. Therefore:

$$g(L_1 \# L_2) = g(F) = g(F_1) + g(F_2) \geq g(L_1) + g(L_2)$$

This concludes our proof.

Note: The incompressibility condition is included because, without it, case (i) surgery of F along λ would produce a disconnected surface with two components with boundaries. Hence, the spanning surface would be disconnected and minimal, which is a plausible concern (since it is the case with split links) so it must be avoided.

Corollary: Genus-1 knots are prime.

Corollary: Knot factorization is finite.

Proof: See Theorem 4.5.1 in Cromwell.