6.1 Loops in Graphs

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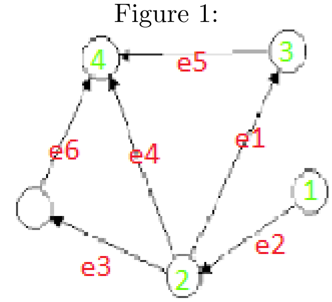
With this lesson, we will expand the tools we have for looking at graphs slightly. The topics should not be to difficult. For the definitions of this lesson, let G be a connected oriented graph with vertex set V and edge set E. By oriented we mean that $\forall$ edge $e \in E$, e is an ordered pair $[v_i, v_j]$ which is oriented from v_i to v_j.

Definition 0.1. Now consider the set S of all formal linear combinations of edges with coefficients taken from the abelian group $(\mathbb{Z}, +)$. We can thus express any element $c \in S$ in the form $c = \sum_{i=1}^{n} \lambda_i e_i$ where n is the number of edges of G, and $\lambda_i \in \mathbb{Z}$. We call these elements of S (written in linear combonation form) **1-chains**.

Any path or circuit in G has a corresponding 1-chain. To find this one-chain, we let λ_i be the integer corresponding to the number of times a given path follows edge e_i. Also, when counting we make sure to consider the orientation of the edge e_i so that we add -1 to λ_i whenever the path follows e_i in the oppositie direction of its orientation. We can observe that a simple path will have a 1-chain in which all λ_i are 0 or ± 1.

Figure 1:



Example 0.2. Try and write the full 1-chain corresponding to the path defined by the vertices of the graph in Figure 1 taken in numerical order.

We now consider combining two one chains using the following rule:

$$\sum_{i=1}^{n} \lambda_i e_i + \sum_{i=1}^{n} \mu_i e_i = \sum_{i=1}^{n} (\lambda_i + \mu_i) e_i$$

With this operation, the set of 1-chains becomes an abelian group, denoted by $C_1(G)$, isomorphic to $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} = \mathbb{Z}^n$.

Definition 0.3. We can follow a similar procedure and define a group of **0-chains** $C_0(G)$, based on formal linear combinations of the vertices of G. Vertices cannot be oriented so there is no convention about $-v$.

To move onto the next topic discussed we will need to look at definition I could not seem to find in Cromwell. It is a definition I found online at http://cims.nyu.edu/ kiryl/teaching/aa/les110703.pdf.

Definition 0.4. Given two groups $(G, *)$ and $(H, \%)$, a function $f : G \longrightarrow H$ is a homomorphism if $f(x * y) = f(x)\% f(y) \quad \forall x, y \in G$.

Definition 0.5. Now we look at a group homomorphism $\partial : C_1(G) \longrightarrow C_0(G)$ known as the **boundary operator** and defined by $\partial([v_i, v_j]) = v_j - v_i$.

Cromwell claims this map is linear, and so we say that:

$$\partial\left(\sum_{i=1}^{n}\lambda_i e_i\right)=\sum_{i=1}^{n}\lambda_i\partial(e_i)$$

Definition 0.6. The kernel (or null space) of this map is a special subgroup of $C_1(G)$). The 1-chains that map to zero have no boundary and are called **1-cycles**. This subgroup of 1-cycles is denoted $Z_1(G)$.

One can observe that the element $0 \in Z_1(G)$, where 0 is the 1-cycle that contains no edges. We call this the trivial 1-cycle. From this information, we can move onto the two results of this section.

Theorem 0.7. *A tree has no non-trivial 1-cycles.*

Proof: Let z be a 1-cycle in a tree G. The proof proceeds by induction on n, the number of edges in G. If n = 1, then

$$0=\partial(z)=\lambda_1\partial(e_1)=\lambda_1 v_2-\lambda_1 v_1.$$

Since there are no other edges to contribute extra terms, we have that $\lambda_1 = 0$. Thus z is trivial.

Suppose now that $n > 1$ and that edge e_n is a leaf connecting vertices v_1 and v_2 where v_1 is the terminal node. Then

$$0=\partial(z)=\partial\left(\sum_{i=1}^{n}\lambda_i e_i\right)=\partial\left(\sum_{i=1}^{n-1}\lambda_i e_i\right)+\lambda_n\partial(e_n).$$

We also have $\pm\partial(e_n) = v_2 - v_1$. Since v_1 is a terminal vertex, there are nos other terms involving it and we deduce that $\lambda_n = 0$. Hence z is a cycle on the tree $G - e_n$, which has $n - 1$ edges. The result follows by induction. □

Definition 0.8. A **basis** for $Z_1(G)$ is a set of cycles such that each 1-cycle in $Z_1(G)$ can be expressed uniquiely as a linear combination of the basis elements. It is in this sense that a graph has a maximum number of independent circuits.

We will now describe a method for constructing a basis for $Z_1(G)$. We will then prove the process does in fact produce a basis.

1. Let T be a spanning tree for$G$.

2. Label the edges of G so that the first r edges are not in the tree: $e_1, ..., e_r \notin T$, and $e_{r+1}, ..., e_n \in T$.

3. For each edge e_i not in T, the graph $T \cup e_i$ contains a uniqe circuit: let $z_i \in Z_1(G)$ be the 1-cycle corresponding to this circuit with the orientation chosen so that the coefficient of e_i is $+1$. This gives a set of r 1-cycles.

Note that r is the rank of the graph and is independent of the tree chosen.

Theorem 0.9. *The set of 1-cycles just constructed forms a basis for $Z_1(G)$.*

Proof: If G is a tree, $r = 0$ and the basis is empty. By the previous theorem, there are no non-trivial 1-cycles, sot the theorem is tre in this case. Now assume that G is not a tree and $r > 0$.

Suppose that $z \in Z_1(G)$ is a 1-cycle. We need to find coefficients $nu_i \in \mathbb{Z}$ such that

$$z = \sum_{i=1}^{r} \nu_i z_i.$$

Since z is a 1-cycle, it is also a 1-chain and we can write it as

$$z = \sum_{i=1}^{n} \lambda_i e_i.$$

The first r of these λ_i are the required coefficients. To see this we show that the difference is zero. Condier

$$z - \sum_{i=1}^{r} \lambda_i z_i.$$

This expression is a cycle since it is a sume of cycles. As a 1-chain, it has coefficient zero on edges $e_1, ..., e_r$ by construction, so it is a 1-cycle in the tree T. A tree has no non-trivial 1-cycles, hence the coefficients on edges $e_{r+1}, ..., e_n$ are also zero.

It remains to show that the coefficients are unique, which Cromwell graciously leaves as an exercise. □