SEIFERT MATRIX

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1. Defining the Seifert Matrix

Recall that oriented surfaces have positive and negative sides. We will be looking at embeddings of connected oriented surfaces in $\mathbb{R}^3$.

Definition 1.1. Let $b : F \times [-1, 1] \to \mathbb{R}^3$ be a homeomorphism with the following properties:

- $b(F \times 0) = F$
- $b(F \times 1)$ lies on the positive side of F

Any subset $X \subset F$ can be lifted out of the surface on either side, so we let

- $X^+ = b(X \times 1)$
- $X^- = b(X \times -1)$

We consider the case when X is a loop in the surface F. Two loops in F could intersect, but if we lift one out of F this cannot happen. In this case, the two loops have a linking number, and we use that to define a map:

$$\Theta : H_1(F) \times H_1(F) \to \mathbb{Z} \tag{1.1}$$

$$(a, b) \to lk(a, b^+) \tag{1.2}$$

We call this the *Seifert pairing* or *linking form* of the embedded surface F. A *Seifert Matrix* is a matrix of the form:

$$M_{i,j} = lk(a_i, a_j^+) \tag{1.3}$$

We will now calculate the Seifert Matrix of a right-handed trefoil.

2. Creating a Link Invariant

Seifert matrices are not link invariants because links can be boundaries of surfaces with different qualities. We do know that two surfaces spanning a link are related by a sequence of piping and compressing, so we look to find a way in which this matrix can be invariant under these operations.

Lemma 2.1. *Let M be a Seifert matrix of a connected surface F, and $\hat{F}$ be a surface made by adding a tube to F. Then there is a basis for $H_1(\hat{F})$*

which contains basis elements of $H_1(F)$ as a subset, and such that the Seifert matrix of $\hat{F}$ has the form

$$\begin{pmatrix} & & & * & 0 \\ & M & & \vdots & \vdots \\ & & & * & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} & & & 0 & 0 \\ & M & & \vdots & \vdots \\ & & & 0 & 0 \\ * & \cdots & * & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

Proof. Suppose we add a tube to the surface, as shown in Figure 6.2 in Cromwell. Let m be the meridian of the tube and l a curve that runs around the tube and back through F. Let the outside of the tube be the positive side of the surface. We see that $lk(m, l^+) = 0 = lk(m, m^+)$ and $lk(l, m^+) = 1$. We can also choose l so that $lk(l, l^+) = 0$. If $lk(l, l^+) = \lambda \neq 0$, we can replace l with $l - \lambda m$ and do the following:

$$\begin{aligned} lk((l-\lambda m),(l-\lambda m)^+) &= lk(l,l^+) - \lambda lk(m,l^+) - \lambda lk(l,m^+) + \lambda^2 lk(m,m^+) \\ &= lk(l,l^+) - \lambda \\ &= 0 \end{aligned}$$

Let $(a_1, \ldots, a_n)$ be a basis for $H_1(F)$. Clearly $lk(a_i, m^+) = lk(m, a_i^+) = 0$ for all i. We do not know how the added tube interacts with other parts of the surface, so we can not know what ways a_i^+ links with l or how a_i links with l^+. Let $\lambda_i = lk(l, a_i^+)$. Therefore we see that adding a tube to F essentially enlarges the matrix M in the following way:

$$\begin{pmatrix} & a_1^+ & \cdots & a_n^+ & l_+ & m_+ \\ a_1 & & & & * & 0 \\ \vdots & & M & & \vdots & \vdots \\ a_n & & & & * & 0 \\ l & \lambda_1 & \cdots & \lambda_n & 0 & 1 \\ m & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

Using the trick as above we can change the basis of $H_1(\hat{F})$ so that each λ_i becomes zero. Replace each a_i with $b_i = a_i - \lambda_i m$. Then

$$lk(l, b_i^+) = lk(l, (a_i - \lambda_i m)^+) = lk(l, a_i^+) - \lambda_i lk(l, m^+) = 0$$

We still have that $lk(b_i, m^+) = lk(m, b_i^+) = lk(b_i, l^+) = 0$. We also have that

$$\begin{aligned} lk(b_i, b_j^+) &= lk((a_i - \lambda_i m), (a_j - \lambda_j m)^+) \\ &= lk(a_i, a_j^+) - \lambda_i lk(m, a_j^+) - \lambda_j lk(a_i, m^+) + \lambda_i \lambda_j lk(m, m^+) \\ &= lk(a_i, a_j^+) \end{aligned}$$

We now have a matrix that looks like the first one stated in the theorem. If we chose the outside of the tube as the negative side, it would look like the other matrix. □

Matrices related by finite sequences of these enlargement and reduction operations, as well as with the congruence transformations $M \to P^T MP$ from a change of basis, are called S-equivalent. We also know that surfaces related by surgery are S-equivalent. This grants us the following theorem.

Theorem 2.2. *Two surfaces that are S-equivalent have S-equivalent Seifert matrices.*

From this, properties of Seifert matrices that are invariant under S-equivalence are also link invariants.

Definition 2.3. The *determinant* of a link L, which we denote $det(L)$, is the absolute value of the determinant of $M + M^T$, where M is any Seifert matrix for L.

Definition 2.4. The *signature* of a link L, which we denote $\sigma(L)$, is the signature of $M + M^T$ where M is any Seifert matrix for L.

Remark 2.5. Recall that any symmetric matrix A with real entries is congruent to a diagonal matrix, i.e. there is an invertible orthogonal matrix P with real entries and determinant equal to ± 1 such that $P^T AP$ has all of its non-zero entries on its diagonal. The signature of a diagonal matrix is the number of positive entries minus the number of negative entries. Two diagonal matrices are congruent if and only if they have the same number of positive, negative, and zero entries. Congruence preserves signature (Sylvester's Theorem).

With this information, we can show that the determinant and signature of a link are well defined.

Theorem 2.6. *Let F be a surface spanning a link L, and M a Seifert matrix constructed from F. Then $det(L)$ and $\sigma(L)$ are link invariants which depend only on L.*

Proof. We want to show that determinant and signature are preserved by congruence transformations and enlargement operations. For congruence transformations, signature is preserved by Sylvester's Theorem. For determinant, recall that $det(P) = \pm 1$.

$$\begin{aligned} det(P^TMP + (P^TMP)^T) &= det(P^T(M + M^T)P) \\ &= det(P^T)det(M + M^T)det(P) \\ &= det(M + M^T) \end{aligned}$$

For enlargement operations described in Lemma (2.1), we consider an enlarged matrix, which has the basic block structure

$$\begin{pmatrix} & & & * & 0 \\ M & + & M^T & \vdots & \vdots \\ & & & * & 0 \\ * & \cdots & * & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

This changes the determinant to $-det(M + M^T)$, so enlargement will not change $det(L)$. In regards to signature, the $*$ values can be changed to zeroes by means of congruence transformations. We then see that the signature of the matrix is the sum of the signatures of the two diagonal blocks. Exercise 6.9.11 $\Rightarrow$ that the 2×2 enlargement block has signature zero. Thus determinant and signature of a link are link invariants. □

Figure 6.6 in Cromwell shows that, unfortunately, some knots have the same signature and determinant.

3. Properties of signature and determinant of a link

Theorem 3.1. *If K is a knot then $det(K)$ is odd and $\sigma(K)$ is even.*

Proof. Let F be an orientable surface spanning K with Seifert matrix M. Lemma (3.2) will show that $det(M - M^T) = 1$ for a knot. Working modulo 2 we have that $M + M^T = M - M^T$, so $det(K) \equiv 1 \bmod 2$. Since the determinant is non-zero, there are no zeroes in the diagonalized form of $M + M^T$. Since K is a knot, there are $2g(F)$ rows and columns in M. Hence the signature is the difference of two even numbers or of two odd numbers. □

Lemma 3.2. *Let M be a Seifert matrix for a surface F. Then $det(M - M^T) =$*

- 1 *if* $|\delta F| = 1$
- 0 *if* $|\delta F| > 1$.

Proof. Given in the book along with completion of exercise 6.9.9. □

Theorem 3.3. *For any link L*

- $det(-L) = det(L) = det(L^*)$
- $\sigma(-L) = \sigma(L)$
- $\sigma(L^*) = -\sigma(L)$

Proof. Let D be a diagram of link L, F be the projection surface constructed from D, and let M be the Seifert matrix of F. If the orientation of L is reversed, then the positive and negative sides of F are changed. The Seifert matrix for $-L$ is the transpose M^T. Hence $M + M^T$ is not changed. L^* is obtained by switching all the crossings in D. This changes all the signs of the linking numbers used in calculating M. Hence the Seifert matrix for L^* is $-M$. Therefore we see that the determinant of the link stays the same, and the signature changes sign. □

Corollary 3.4. *An amphicheiral link has signature zero.*

Remark 3.5. This is a one-sided implication, as knot 6_1 is a cheiral knot and has signature zero.

Theorem 3.6. *If $L_1 \sqcup L_2$ is a split link then*

- $det(L_1 \sqcup L_2) = 0$
- $\sigma(L_1 \sqcup L_2) = \sigma(L_1) + \sigma(L_2)$

Proof. Let F_i be an orientable surface spanning L_i, and M_i the Seifert matrix. A connected surface F that spans L can be formed by piping F_1 and F_2 together. To create a basis for $H_1(F)$ we take the union of the two bases for F_1 and F_2 with a meridian m of the piping tube. Now $lk(a_j, m^+) = lk(m, a_j^+) = 0$ for any loop $a_j \in H_1(F_i)$. Thus a Seifert matrix for F has the form

$$\begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Clearly the determinant and signature are as said in the theorem. □

Theorem 3.7. *If a link can be factorized as $L_1 \# L_2$, then*

- $det(L_1 \# L_2) = det(L_1)det(L_2)$
- $\sigma(L_1 \# L_2) = \sigma(L_1) + \sigma(L_2)$

Proof. Let F_i be an orientable surface spanning L_i with Seifert matrix M_i. A connected surface F spanning L can be created by inserting a rectangular disc R such that $R \cap F_i = \delta R \cap \delta F_i = \alpha_i$ where α_i is a single arc and α_1 and $\alpha)_2$ are on opposite sides of the rectangle. We take the union of the bases for F_1 and F_2 as the basis for F, which gives us a Seifert matrix for F that looks like

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

□

Remark 3.8. Notice that some non-trivial links have have determinant 1. Therefore the previous theorem does not help detect prime links.

Theorem 3.9. *If S is a satellite knot constructed from pattern P with companion C, framing zero, and winding number n then $det(S) =$*

- *$det(P)$ if n is even*
- *$det(P)det(C)$ if n is odd*

Proof. The proof is two pages long, and is in Cromwell. □

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