Alexander Polynomial and Genus & Skein Relation
7.2 & 7.3

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Definition: Breadth

The **breadth** of a polynomial is the difference between its highest and lowest degrees.

Theorem: The genus of a non-split link, L, is bounded below by the breadth of the Alexander polynomial:

$$2g(L) + \mu(L) - 1 \geq \frac{1}{2} \text{ breadth } \Delta_L(x).$$

Proof: Let F be a connected, minimal-genus, spanning surface for L and let $r = 2g + \mu - 1$. A basis for $H_1(F)$ has r generators, so the Seifert matrix for F is an $r \times r$ square matrix. Thus, the largest possible degree for x is r and the smallest degree is $-r$. Hence, breadth $\Delta_L(x) \leq 2r$. Therefore:

$$\begin{aligned} 2g(F) &= 2 - \chi(F) - \mu(F) \\ &= 2r - \mu(L) + 1 \\ &\geq \frac{1}{2} \text{ breadth } \Delta_L(x) - \mu(L) + 1. \end{aligned}$$

Q.E.D.

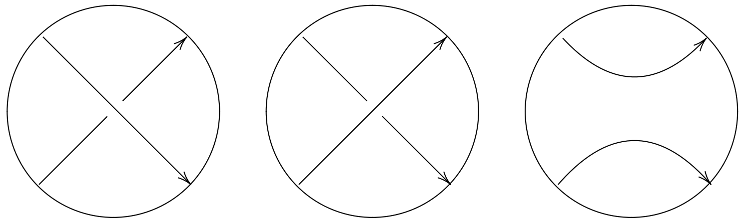
This theorem is powerful enough to establish the genus for all knots up to 10 crossings. However, there are 11-crossing knots for which this fails. For example, if K is the Kinoshita-Terasaka knot (seen on page 98 of Cromwell), then $c(K) = 11$ and $\Delta(K) = 1$. This knot is non-trivial, so its genus is at least 1 (in fact, $g(K) = 2$).

However, for a knot such as 10_{165} (seen on page 140 of Cromwell), we check that:

$$\Delta(10_{165}) = 3x^4 - 11x^2 + 17 - 11x^{-2} + 3x^{-4}.$$

So, while the projection surface of any diagram of 10_{165} has genus 3, the Alexander polynomial gives us a bound of 2. In fact, 10_{165} does have genus 2 (since loop b in Cromwell bounds a compressing disc for the projection surface) so the Alexander polynomial may be used to conclusively determine the genus of a link with ≤ 10 crossings.

Returning to some topics from Jeff's lecture on 07.13, we now define relations between diagrams which are locally different in the following manner:



D_+ D_- D_0

A localized change from D_+ to D_- or vice versa is called *switching* a crossing.
A localized change from D_+ or D_- to D_0 is called *smoothing* a crossing.

We note that these local substitution operations can (and often do) change the link type. As well, smoothing a crossing always increases or decreases the number of components by 1.

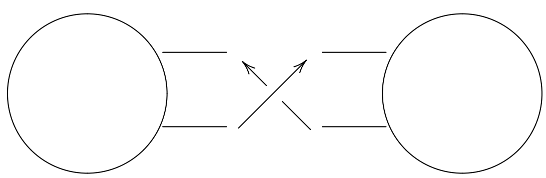
Theorem: If three oriented links, L_+, L_- and L_0, have diagrams, D_+, D_- and D_0, which differ only in a small neighborhood as shown in the above figure, then:

$$\Delta(L_+) - \Delta(L_-) = (x^{-1} - x)\Delta(L_0)$$

Proof: Let F_+, F_- and F_0 be the projection surfaces constructed from D_+, D_- and D_0, respectively. Let M_0 be the Seifert matrix of F_0.

If D_+ is a disconnected diagram, then D_- and D_0 must also be disconnected. This implies that L_+, L_- and L_0 are all split links. So, from a theorem in Fresh B's lecture on 07.14, the relation holds (since all Δs would be 0). The same argument applies when D_- is a disconnected diagram.

Suppose that D_0 is a disconnected diagram but that D_+ and D_- are connected. Then L_0 is a split link and the links, L_+ and L_-, are isotopic (the right side of the following diagram may be turned over). Thus, the relation holds again.



In the remaining case, all the diagrams, D_+, D_- and D_0, are connected. Let $(a_1, \dots, a_n)$ be a basis for $H_1(F_0)$. Each loop, a_i, is also a subset of F_+ and F_-. A basis for $H_1(F_+)$ can be completed by adding a loop, b, which passes once through the twisted band in the tangle and back through the rest of the surface. The Seifert matrix, M_+, for F_+ has the form:

	a_1^+	$\dots$	a_n^+	b^+
a_1				v_1
$\vdots$		M_0		$\vdots$
a_n				v_n
b	λ_1	$\dots$	λ_n	β

Using the same loop, b, in F_- gives a Seifert matrix M_- for F_- that is identical to M_+ except in the bottom-right corner: in F_-, the linking number is: $lk(b, b^+) = \beta + 1$.

Expanding the determinants, $det(xM_+ - x^{-1}M_+^\top)$ and $det(xM_- - x^{-1}M_-^\top)$ about the last column and subtracting terms, we see that almost everything cancels and we are left with:

$$\begin{aligned}\Delta(L_+) - \Delta(L_-) &= \beta(x - x^{-1})det(xM_0 - x^{-1}M_0^\top) - (\beta + 1)(x - x^{-1})det(xM_0 - x^{-1}M_0^\top) \\ &= -(x - x^{-1})det(xM_0 - x^{-1}M_0^\top) \\ &= (x^{-1} - x)\Delta(L_0)\end{aligned}$$

Q.E.D.

Note: This relationship between three links with local differences is an example of a *skein relation*. These relationships only depend on link types, not on the particular diagrams.