Topic: Khovanov Homology

In 2000 Khovanov [**?**] realized that the Jones polynomial could be recovered as the Euler characteristic of a certain (co)chain complex constructed from link diagrams. The goal of this note is to describe this chain complex and demonstrate how it recovers the Jones polynomial. We follow closely the notation and approach of [**?**].

Contents

1 Preliminaries

1.1 Tensor Products

Given two vector spaces V and W, there is a way of forming a new vector space $V \otimes W$, called the tensor product of V and W, which satisfies the property that if $\{v_i\}$ is a basis for V and $\{w_j\}$ is a basis for W, then $\{v_i \otimes w_j\}$ is a basis for $V \otimes W$. Hence if both V and W are finite-dimensional, then the dimension is multiplicative: $\dim(V \otimes W) = (\dim V)(\dim W)$. Moreover, the vector space $V \otimes W$ will satisfy the following universal property: There is a bilinear map $\phi : V \times W \to V \otimes W$ such that if Z is any other vector space and $f : V \times W \to Z$ is a bi-linear map, then there is a unique linear map $g : V \otimes W \to Z$ making the diagram below commute

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[d, "\phi"'] & Z \\ V \otimes W \arrow[ur, "g"'] & \end{tikzcd}$$

These two properties motivate, at least, the notation $\otimes$.

Let V and W be vector spaces. Form the vector space $F(V \times W)$ with basis consisting of all points (v, w) of $V \times W$. Note that this vector space $F(V \times W)$ is very large and generally infinite-dimensional. Let $R(V, W)$ denote the subspace of $F(V \times W)$ generated by all elements of the form

$$\begin{aligned} (v, w_1 + w_2) - (v, w_1) - (v, w_2) \\ (v_1 + v_2, w) - (v_1, w) - (v_2, w) \\ (\lambda v, w) - \lambda(v, w) \\ (v, \lambda w) - \lambda(v, w) \end{aligned}$$

for $v, v_i \in V, w, w_i \in W$ and λ a scalar. Then define the **tensor product of** V **and** W, denoted $V \otimes W$, to be the quotient space $F(V \times W)/R(V, W)$.

For $v \in V$ and $w \in W$, we let $v \otimes w$ denote the equivalence class of (v, w) in the quotient space $V \otimes W$, in other words, $v \otimes w = [(v, w)]$. Note that we have constructed the quotient space so that the following relations hold

$$
\begin{aligned}
v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2 \\
(v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w \\
(\lambda v) \otimes w &= \lambda(v \otimes w) \\
v \otimes (\lambda w) &= \lambda(v \otimes w).
\end{aligned}
$$

Also note that elements of the quotient space $V \otimes W$ take the form of finite sums of simple tensors

$$\sum_{i=1}^{n} v_i \otimes w_i.$$

In particular, note carefully that not all elements of $V \otimes W$ are expressible as simple tensors themselves, but rather as finite sums of such simple tensors.

It is routine to check that the desired properties of $V \otimes W$ are satisfied, and we leave this as an exercise.

Exercise 1.1.

(i) Check that if $\{v_i\}$ is a basis for V and $\{w_j\}$ is a basis for W, then $\{v_i \otimes w_j\}$ is a basis for $V \otimes W$.

(ii) Check that $V \otimes W$ satisfies the universal property listed at the beginning of the section.

1.2 Graded vector spaces

A vector space V is said to be **$\mathbb{Z}$-graded** if V admits a decomposition of the form

$$V = \bigoplus_{m \in \mathbb{Z}} V_m.$$

Vectors in V_m are said to be of **degree** m. The **graded dimension of** V is the formal power series in the variable q given by

$$\mathrm{qdim}(V) = \sum_{m \in \mathbb{Z}} q^m \dim V_m.$$

If $V = \bigoplus_m V_m$ is a graded vector space, let $V\{\ell\}$ denote the same vector space with a shifted grading so that $V\{\ell\}_m = V_{m-\ell}$. This operation has the effect that the graded dimension satisfies

$$\mathrm{qdim}(V\{\ell\}) = q^\ell \mathrm{qdim}(V).$$

If V and W are graded vector spaces and $f : V \to W$ is a linear map satisfying $f(V_m) \subset W_{m+d}$ for some integer d, then f is said to be **homogeneous of degree** d.

A co-chain complex $(C^\bullet, d)$ is a sequence of vector spaces $\{C^\bullet\}$ together with linear maps $d^n : C^n \to C^{n+1}$

such that $d^2 = 0$. For a co-chain complex, the n-th cohomology group is well-defined and given by

$$H^n(C^\bullet) = \frac{\ker(d^n : C^n \to C^{n+1})}{\mathrm{im}(d^{n-1} : C^{n-1} \to C^n)}.$$

If $(C^\bullet, d)$ is a chain complex of (possibly graded) vector spaces, then we can also define a shifted chain complex $C^\bullet[\ell]$ so that $C^\bullet[\ell] = C^{\ell-n}$ with the differentials shifted as well.

The **graded Euler characteristic** of a graded chain complex $C^\bullet$ is defined to be the alternating sum of its cohomology groups:

$$\chi_q(C^\bullet) := \sum_{n\in\mathbb{Z}} (-1)^n \mathrm{qdim} H^n(C^\bullet).$$

In the case where all the vector spaces C^n are finite dimensional, only finitely many of them are nonzero, and the differentials have degree 0, one can show that

$$\chi_q(C^\bullet) = \sum_{n\in\mathbb{Z}} (-1)^n \mathrm{qdim}(C^n).$$

1.3 Jones polynomial

It will be convenient to describe another approach to the Jones polynomial that will shift the degrees of the variable a bit. However, recall first our approach from class:

There is a way of associating a bracket polynomial $\langle |D| \rangle(A)$ in the variable A to an unoriented diagram $|D|$. The normalized bracket polynomial is then a polynomial constructed from an oriented diagram D by declaring

$$\tilde{V}_D(A) := (-A^3)^{w(D)} \langle |D| \rangle(A),$$

where $w(D)$ denotes the writhe of D. Note that $V_D(A)$ is allowed to have negative powers of the variable A, and hence is a Laurent polynomial in A. The Jones polynomial is then constructed from normalized bracket polynomial by a substitution of variables. In particular, the Jones polynomial $V_D(t)$ is defined by

$$V_D(t) = \tilde{V}_D(t^{-1/4}).$$

This is not a polynomial in strict terms, since it will have non-integer powers of t. Instead it it is a Laurent polynomial in the variable $t^{1/2}$.

Now instead let us proceed with different conventions. Let D be an oriented link diagram, let n_+ denote the number of positive crossings and n_- the number of negative crossings. Define a new bracket polynomial, which we will denote by $\langle D \rangle$ by the rules

(i) $\langle \bigcirc \rangle = 1$

(ii) $\langle D \sqcup \bigcirc \rangle = (q - q^{-1}) \langle D \rangle$

(iii) $\langle D_+\rangle = \langle D_0\rangle - q\langle D_\infty\rangle$.

Then define the Jones polynomial by

$$J(D) = (-1)^{n_-}q^{n_+-2n_-}\langle D\rangle$$

To compute the Jones polynomial, one can proceed in the following fashion. Let χ denote the set of crossings of D. For a crossing D_+, let D_0 denote the 0-smoothing of D_+ and D_∞ the 1-smoothing. In this way, each $\alpha \in \{0,1\}^\chi$ corresponds to a vertex of an $n = n_+ + n_-$ dimensional cube, which has smoothed each crossing according to α to produce a complete smoothing diagram D_α. If $h(\alpha)$ denotes the height of α, then

$$\langle D\rangle = \sum_{\alpha\in\{0,1\}^\chi} (-1)^{h(\alpha)}q^{h(\alpha)}(q+q^{-1})^{\mu(D_\alpha)}$$

2 Khovanov's chain complex

Following [?], we construct a chain complex associated to a link diagram D whose graded Euler characteristic is the Jones polynomial.

Let D be a link diagram with $\chi, n = n_+ + n_-$ as before. Let V be a graded vector space with two basis elements $v_\pm$ which have degrees ± 1 respectively, and hence $\mathrm{qdim}(V) = q + q^{-1}$. For each complete smoothing D_α ($\alpha \in \{0,1\}^\chi$), we define a vector space

$$V_\alpha(D) = V^{\otimes \mu(D_\alpha)}\{h(\alpha)\}.$$

Note that

$$\mathrm{qdim}(V_\alpha(D)) = q^{h(\alpha)}(q+q^{-1})^{\mu(D_\alpha)}.$$

Then define an auxilary chain complex $\tilde{C}^\bullet(D)$ by declaring $\tilde{C}^r(D)$ to be the sum of all vector spaces of height r:

$$\tilde{C}^r(D) := \bigoplus_{h(\alpha)=r} V_\alpha(D).$$

Omitting the point of the differentials for now, we define the chain complex $C^\bullet(D)$ by shifting accordingly:

$$C^\bullet(D) := \tilde{C}^\bullet(D)[-n_-]\{n_+ - 2n_-\}$$

It now follows almost immediately from the definitions that the following theorem holds.

Theorem 2.1. *The graded Euler characteristic of $C^\bullet(D)$ is the Jones polynomial of D.*

To conclude, we indicate the differential ∂ maps of the complex $C^\bullet(D)$. The additional structure made possible by these maps means that $C^\bullet(D)$ is in fact a chain complex of vector spaces.

The chain complex will be formed by defining boundary maps along the "edges" of the n-dimensional cube $\{0,1\}^{\chi}$. In particular, if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a vertex of the cube and α' is another vertex obtained from α by changing the kth entry of α from a 0 to a 1, then we will define a linear map

$$d^{\alpha_1 \cdots \alpha_{k-1} \star \alpha_{k+1} \cdots \alpha_n} : V_\alpha(D) \to V_{\alpha'}(D).$$

For notation, set $\xi = \alpha_1 \cdots \alpha_{k-1} \star \alpha_{k+1} \cdots \alpha_n \in \{0, 1, \star\}^n$.

The smoothing $D_{\alpha'}$ is obtained from the smoothing D_α by one of two procedures:

(i) Two of the loops of D_α merge to form one loop in $D_{\alpha'}$

(ii) One of the loops of D_α splits to become two loops in $D_{\alpha'}$.

In either case, we let d^ξ act as the identity on the factors of V corresponding to loops that are not changed. To define what d^ξ does on the loops that participate, we consider the two cases:

(i) In case (i), we need a map $m : V \otimes V \to V$. We let m denote the unique linear map satisfying

$$\begin{aligned} v_+ \otimes v_- &\mapsto v_- \\ v_+ \otimes v_+ &\mapsto v_+ \\ v_- \otimes v_+ &\mapsto v_- \\ v_- \otimes v_- &\mapsto 0. \end{aligned}$$

Note that m does indeed define a unique linear map because we have specified where m sends a basis of $V \otimes V$. Note that m is a map of degree -1.

(ii) In case (ii), we need a map $\Delta : V \to V \otimes V$. We let Δ denote the unique linear map satisfying

$$\begin{aligned} v_+ &\mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- &\mapsto v_- \otimes v_-. \end{aligned}$$

Note that Δ is a map of degree -1.

It then follows that the maps d^ξ for edges ξ of the form above make the edges of the n-dimensional cube into a commutative diagram. However, in order to form a chain complex (with $d^2 = 0$) we actually need all square faces to anti-commute. To do this, we modify the maps d_ξ by multiplying by

$$(-1)^\xi := (-1)^{\sum_{i<k} \xi_i}$$

where k is the location of where $\star$ is in ξ. In this way, the new maps $(-1)^\xi d^\xi$ make the edges of all square faces anti-commute.

We then define coboundary maps $d^r : C^r(D) \to C^r(D)$ by letting

$$d^r = \sum_{|\xi|=r} (-1)^{\xi} d^{\xi}$$

where $|\xi|$ denotes the height of the tail of ξ.

References

[K] M. Khovanov, A categorification of the Jones polynomial, *Duke Math. J.* 101 (2000), no. 3, 359-426.

[B] D. Bar-Natan, On Khovanov's categorification of the Jones polynomial, *Algebr. Geom. Topol.* 2 (2002) 337-370.