NOTES ON MANIFOLDS

JACK BURKART

1. SOME MORE TOPOLOGY

1.1. **Seperation Axioms.** Distance and the notion closeness are very important ideas at the heart of analysis in $\mathbb{R}^n$. If we wish to extend basic ideas of analysis to general topoligical spaces, we'd like to have some way to detect these ideas. It turns out that the so-called seperation axioms do a good job of encoding much of this information.

Definition 1.1. Let X be a topological space.

(1) We say that X is **Hausdorff** if for every distinct $x, y \in X$, there exists disjoint open sets $U, V \subset X$ so that $x \in U$ and $y \in V$
(2) We say that X is **Regular** if for each point x and every closed set E not containing X, there exists disjoint open sets $U, V \subset X$ so that $x \in U$ and $E \subset V$.
(3) We say that X is **Normal** if for every disjoint pair of closed sets $E, F \subset X$, there exists disjoint open sets $U, V \subset X$ so that $E \subset U$ and $F \subset V$.

There are weaker seperation axioms, but these are the most important ones. Clearly, if the one point sets are closed, then normal spaces are regular spaces and regular spaces are Hausdorff.

1.2. **Countability Axioms.**

Definition 1.2. A topological space X has a **countable basis at** x if there is a countable collection $\mathcal{B}$ of open sets containing x so that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. If X has a countable basis at each point $x \in X$, we call X **first-countable**.

Definition 1.3. A topological space X is called **second-countable** if there exists a countable basis for its topology.

All second-countable spaces turn out to be first-countable as well if you fiddle with the definitions. It's important to remark that second-countability is a really strong condition for topological spaces to satisfy. Whereas most of the topological spaces considered in analysis are Hausdorff (or one of the stronger conditions) many are not second-countable.

Since we'll soon be considering manifolds, the most important conditions for us are Hausdorff and second-countability. These will make sure that our manifolds look like what we want and don't exhibit any exotic behavior. The second-countability axiom, in particular, is very important for the construction of special functions on manifolds where this hypothesis is needed, but fortunately does not come up too often (if at all really) outside of those constructions.

1.3. **Compactness.** Finally, we desire spaces where we can take local properties of functions, such as continuity and boundedness, and extend this properties to a more global setting. Compactness is one such property.

Definition 1.4. Let X be a topological space. An **open cover** of any subset $K \subset X$ is a collection of open subsets $\{U_\alpha\}$ of X such that

$$K \subset \bigcup U_\alpha.$$

We say K is **compact** if every open cover admits a finite subcover. More precisely, if $\{U_\alpha\}$ is an open cover of K, then there are open sets in the open cover $U_1, \dots U_n$ such that

$$K \subset \bigcup_{i=1}^{n} U_i.$$

The next example is actually a theorem.

Theorem 1.5. *Consider* $\mathbb{R}$ *with the standard topology. The unit interval* $[0,1]$ *is compact as a subset of* $\mathbb{R}$.

Proof. Let $\{U_\alpha\}$ be *any* open cover of $[0,1]$. We have to find a finite subcover. Since we really know nothing about the cover, it's hopeless to construct one explicitly, so we'll have to use an indirect method. Recall that the least upper bound property of the real numbers states that any non-empty subset of the real numbers that has an upper bound has a *least* upper bound, a bound that is smaller than all other upper bounds.

We need a set to apply the least uper bound property to; we choose the set

$$A = \big\{x \in [0,1] : [0,x] \text{ is covered by finitely many members of } \{U_\alpha\}\big\}.$$

To apply the least upper bound property, we must show this set is non-empty and has an upper bound. By definition, 1 is an upper bound, and it is obvious that 0 must be in A. Therefore, the set A has a least upper bound, say γ.

Suppose for the sake of contradiction that $\gamma < 1$. We must have that $A = [0,\gamma)$ or $A = [0,\gamma]$. The first case is actually impossible. Cover $[0,\gamma)$ with finitely many open sets from the open cover. Since we're working with an open cover of $[0,1]$, there must be one other open set containing γ. If we add that to the finite collection, it's still finite. So $A = [0,\gamma]$. Let U_0 be the open set that contains γ. Since the set is open, there is an $\epsilon > 0$ so that $B_\epsilon(\gamma) \subset U_0$. In particular, $\gamma + \epsilon/2 \in U_0$. But γ was supposed to be an upper bound for A, so this is impossible. We must have $\gamma = 1$, which proves the claim. □

2. SMOOTH MANIFOLDS

Loosely put, smooth manifolds are spaces that locally look flat. Consider the Earth. We're small enough so that if you looked outside, it would appear as though we were living on some plane. In mathematical language, S^2 locally looks like $\mathbb{R}^2$ if you zoom in enough. However, this idea of being locally like some flat Euclidean space is not the whole story.

Definition 2.1. Let X be a topoligical space. A **n dimensional coordinate chart** at x is a pair (U,φ) such that

(1) U is an open subset of X containing x.

(2) φ is a function $\varphi : U \to \mathbb{R}^n$ for some n so that φ is a homeomorphism onto $\varphi(U) \subset \mathbb{R}^n$.

If there is a coordinate chart at each point $x \in X$ and X is Hausdorff, we say X is **locally Euclidean of dimension n**

This describes what we stated at the start of this section. To get to what we need for a smooth manifold, we're going to require some compatibility between these coordinate charts for when they overlap.

Definition 2.2. Let X be locally Euclidean of dimension n. A **smooth atlas** $\mathcal{A}$ on X is a collection $\{(U_\alpha, \varphi_\alpha)\}$ of coordinate charts satisfying

(1) The U_α's cover X, that is
$$\bigcup U_\alpha = X$$

(2) For every pair (α, β), if $U_\alpha \cap U_\beta \neq \emptyset$, then the map
$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$
is smooth as a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. To be brief, we often describe the above by saying the **transition functions** are all smooth.

One immediately runs into an annoying technical point that there are in principle many possible choices of smooth atlases for a locally Euclidean space X. We won't worry about this too much, just assume that we are working with the biggest possible atlas. In other words, if some other chart (V, ψ) is smoothly compatible (satisfies condition (2) above) with all the other charts in $\mathcal{A}$, then (V, ψ) is actually in $\mathcal{A}$ as well.

Now that we understand a smooth atlas, we can define a smooth manifold.

Definition 2.3. An m**-dimensional smooth manifold** M is a second-countable, locally Euclidean space of dimension m equipped with a smooth atlas.

We have to check *a lot* of things here; it took a lot of work for us to turn the manifold definition into something that fits in one line of text. First, we need to check that our underlying topological space M is second-countable and Hausdorf. Most of the time, this is fortunately immediate, since subspaces of Hausdorff/second-countable spaces are Hausdorff/second-countable spaces, and most of the manifolds we'll consider are subspaces of $\mathbb{R}^n$, which has these properties. Next, we need to construct coordinate charts that cover the whole space M, which is a process that varies from manifold to manifold. Finally, we need to check that the transition functions are all smooth!

E-mail address: `jburkar1@nd.edu`