

The material in these notes is motivated by Chapter 2 of Cromwell. A source I used for smooth manifolds is do Carmo's *Riemannian Geometry*. Ideas of multivariable Calculus can be found in Apostol or Rudin. General exposition of other material can be found on Wikipedia.

6.1 Orientation

Let $B(\mathbb{R}^n)$ denote the collection of ordered bases of \mathbb{R}^n . For any pair of bases $e = (e_1, \dots, e_n)$ and $f = (f_1, \dots, f_n)$, there is a unique invertible linear transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $Ae_i = f_i$. Define an equivalence relation on $B(\mathbb{R}^n)$ by $e \sim f$ if and only if the unique change of basis matrix A satisfying $Ae_i = f_i$ has positive determinant. An **orientation** on \mathbb{R}^n is a choice of equivalence class under this equivalence relation.

Exercise 6.1. Check that the above relation defines an equivalence relation.

Proposition 6.2. *There are two orientations on \mathbb{R}^n .*

Proof. There is at least one orientation corresponding to the standard basis $e = (e_1, \dots, e_n)$. The basis $f = (-e_1, e_2, \dots, e_n)$ provides another orientation, since the change of basis matrix from this to the standard basis has determinant -1 . Let $g = (g_1, \dots, g_n)$ be an arbitrary ordered basis. Suppose that g is not equivalent to e nor f . Let A denote the change of basis matrix satisfying $Ae_i = g_i$ and B the matrix satisfying $Bf_i = g_i$. Since g is not equivalent to either e or f , we have $\det A < 0$ and $\det B < 0$. It follows that $\det B^{-1}A = (\det B)^{-1} \det(A) > 0$. This implies that e is equivalent to f , which is a contradiction. \square

Let U be a connected subset of \mathbb{R}^n and let $\phi : U \rightarrow \mathbb{R}^m$ be a smooth map. Recall that the **derivative of ϕ at $x \in U$** is the linear map $d\phi_x : \mathbb{R}^n \rightarrow \mathbb{R}^m$ whose corresponding matrix with respect to the standard bases is

$$\begin{bmatrix} \frac{\partial \phi_1}{\partial x_1}(x) & \frac{\partial \phi_1}{\partial x_2}(x) & \cdots & \frac{\partial \phi_1}{\partial x_n}(x) \\ \frac{\partial \phi_2}{\partial x_1}(x) & \frac{\partial \phi_2}{\partial x_2}(x) & \cdots & \frac{\partial \phi_2}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_m}{\partial x_1}(x) & \frac{\partial \phi_m}{\partial x_2}(x) & \cdots & \frac{\partial \phi_m}{\partial x_n}(x) \end{bmatrix}$$

where $\phi = (\phi_1, \dots, \phi_m)$.

The chain rule states that for smooth $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\psi : \mathbb{R}^m \rightarrow \mathbb{R}^k$ and a point $x \in \mathbb{R}^n$, the derivative of the composition $\psi \circ \phi$ at x is $d(\psi \circ \phi)_x = d\psi_{\phi(x)} \circ d\phi_x$.

Exercise 6.3. Show that if $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a smooth map with a smooth inverse $\psi : \mathbb{R}^m \rightarrow \mathbb{R}^n$, then $d\phi_x$ is invertible for each $x \in \mathbb{R}^n$ and $m = n$.

Let U be a path connected subset of \mathbb{R}^n and suppose that $\phi : U \rightarrow \mathbb{R}^n$ is smooth with a smooth inverse. We say that ϕ is **orientation preserving** if the matrix $d\phi_x$ has positive determinant for some point (and hence all points) $x \in U$. We say that ϕ is **orientation reversing** if it is not orientation preserving.

Exercise 6.4. Show that in the above situation, if $d\phi_x$ has positive determinant for some point $x \in U$, then $d\phi_y$ has positive determinant for all other points $y \in U$.

As an example, on \mathbb{R}^3 , then identity map is orientation preserving. The reflection $r : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ about the origin given by $r(x, y, z) = (-x, -y, -z)$ is orientation reversing.

We may now talk about oriented manifolds. However, we will only work with orientations on *smooth manifolds*, which will be defined now.

An **n -dimensional coordinate chart** on an n -dimensional manifold M is a pair (U, ϕ) where U is an open subset of M and $\phi : U \rightarrow \mathbb{R}^n$ is a continuous map taking U homeomorphically onto $\phi(U)$. A **smooth atlas** for M is a collection $\{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ of coordinate charts such that

- The collection $\{U_\alpha : \alpha \in A\}$ covers M
- Whenever $U_\alpha \cap U_\beta \neq \emptyset$, the **transition map** $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$ is a smooth map of open subsets of \mathbb{R}^n .

(Note that the transition maps are already assumed to be homeomorphisms, as they are compositions of homeomorphisms.) We may define an equivalence relation on smooth atlases by declaring two atlases to be equivalent if and only if their union is a smooth atlas. A **smooth structure** on M is then a choice of an equivalence class of smooth atlases. A **smooth manifold of dimension n** is then a topological manifold of dimension n together with a choice of smooth structure.

Morally the above definition says that smooth manifold admits a covering by open subsets of \mathbb{R}^n that are patched together in a smooth way. From this interpretation, it is intuitively clear that the following examples of manifolds are smooth

- any open subset U of \mathbb{R}^n
- the unit sphere S^n
- the torus T .

We leave it to the reader to check the details of these assertions.

We may now introduce the notion of an oriented smooth manifold. Say that M is an n -dimensional manifold with smooth structure $\{(U_\alpha, \phi_\alpha)\}$. We say that M is **oriented** if all the transition functions $\phi_\beta \circ \phi_\alpha^{-1}$ are orientation preserving.

In particular \mathbb{R}^n itself is canonically oriented, since a smooth atlas consists of the identity function, which is orientation preserving. Any open subset U of \mathbb{R}^n is also oriented.

One may show that the smooth manifolds S^n and T can be oriented, but I will not try to do so here.

For surfaces, it can be shown that choosing an orientation amounts to choosing two sides to the surface. Hence, we sometimes say that orientable surfaces are two-sided. From this viewpoint, it is clear that the unit sphere S^2 and the torus T can be oriented. However, there are examples of smooth surfaces that do

not admit any smooth structure that is oriented. Such surfaces are called **non-orientable**. For example, the Möbius band is not orientable: it only has one side.

6.2 Surfaces

In homework, we saw that there are at least two non-homeomorphic one-dimensional manifolds, namely, the real line \mathbb{R} (or the unit interval $(0, 1)$) and the unit circle S^1 . One can show that, in fact, these are the only one-dimensional manifolds, and hence we understand all examples of one-dimensional manifolds!

Now if we study one-dimensional manifolds with boundary, the list increases in size. We can for example have the manifold $[0, 1)$, which has one point on its boundary, and $[0, 1]$, which has two points on its boundary. These will not be homeomorphic to each other, nor to $(0, 1)$ or S^1 , and hence there are at least four different one-dimensional manifolds with boundary. One may in fact show that these are all the possibilities.

When we increase the dimension, the story becomes a lot more complicated. In fact, there are infinitely many non-homeomorphic two-dimensional manifolds, or **surfaces**, as we will show.

We are already familiar with the unit sphere S^2 and the torus T . Moreover, we already know that any open subset U of \mathbb{R}^2 acquires the structure of a two-dimensional manifold in the usual way. Our goal now is to construct several other examples of surfaces by using quotient spaces.

6.2.1 Connected sum

Let X and Y be surfaces. We can form a new surface, called the connected sum of X and Y , denoted $X \# Y$, which is morally obtained by cutting out unit discs in X and Y and gluing the remaining two pieces together along their boundaries. To make this precise, we proceed in the following manner.

Exercise 6.5. For topological spaces X_1 and X_2 , the disjoint union $X_1 \sqcup X_2$ enjoys the structure of a topological space where the open sets are those of the form $U_1 \sqcup U_2$ with U_i open in X_i . Moreover, if X_1 and X_2 are surfaces, then so is $X_1 \sqcup X_2$.

For our surface X , there is an open set $B \subset X$ that is homeomorphic to the unit ball in \mathbb{R}^2 . There is similarly an open subset $C \subset Y$, which is homeomorphic to the unit ball. The space $(X \setminus B) \sqcup (Y \setminus C)$ enjoys the structure of a surface with boundary. If X and Y have no boundary, then there are two boundary components $\partial(X \setminus B)$ and $\partial(Y \setminus C)$ of $(X \setminus B) \sqcup (Y \setminus C)$. These two boundary components are in correspondence with the boundary of the unit disc, so we have homeomorphisms

$$\phi : \partial(X \setminus B) \rightarrow S^1$$

$$\psi : \partial(Y \setminus C) \rightarrow S^1$$

and hence we have a homeomorphism $\psi^{-1} \circ \phi : \partial(X \setminus B) \simeq \partial(Y \setminus C)$. We wish to glue the two components along these boundaries. So define an equivalence relation \sim on $(X \setminus B) \sqcup (Y \setminus C)$ by $x \sim y$ if and only if

$\psi^{-1} \circ \phi(x) = y$. Let $X \# Y$ denote the resulting quotient space

$$X \# Y = [(X \setminus B) \sqcup (Y \setminus C)] / \sim .$$

This certainly acquires the structure of a topological space. Moreover, it can be shown to have the structure of a surface. Finally, it can be shown that the homeomorphism class of $X \# Y$ does not depend on the choice of balls B and C .

6.2.2 Klein bottle and projective space

We often use a type of polygon representation for surfaces.

For example, we represent the torus by a box, where we have identified the two edges in a certain manner, namely, the left and right edges are identified in the same way as are the top and bottom. The resulting quotient space that this picture represents is homeomorphic to the torus.

We might ask what happens if we reverse the orientation of one of the identifications of the pairs of edges. The resulting space we obtain is called the **Klein bottle**, denoted K , and it can be shown to be different than the torus and the sphere.

We may represent the sphere by a single point with a single loop. All points of the loop are identified to a single point, thus giving a space homeomorphic to the two-sphere S^2 .

Finally, there is the projective space, which can be represented by two arcs connecting two points, identified in opposite directions. The resulting space is called projective space, and it is not homeomorphic to S^2 , T , or K .

6.3 Euler characteristic and genus

For surface X , a **triangulation** X^Δ of X is a way of cutting X into a finite number of pieces each of which is homeomorphic to a triangle. We also require some assumptions about this cutting, so that this definition can be made much more precise, but we hope that the geometric picture is clear.

From such a triangulation X^Δ , we can obtain a number $\chi(X^\Delta)$, called the Euler characteristic of X^Δ , which is defined as

$$\chi(X^\Delta) = \#V - \#E + \#F$$

where $\#V$ is the number of vertices, $\#E$ is the number of edges, and $\#F$ is the number of faces in the triangulation. One may show that this number does not depend on the choice of triangulation, and hence defines an invariant of the surface X , called the **Euler characteristic**, denoted $\chi(X)$.

To see why this number does not depend on the choice of triangulation, consider the following argument. For any two triangulations X_1^Δ and X_2^Δ , there is a third refining both X_1^Δ and X_2^Δ . So it suffices to show

that if X^Δ is a refinement of X_1^Δ , then

$$\#V - \#E + \#F = \#V_1 - \#E_1 + \#F_1.$$

For this, it suffices to take X^Δ to be a refinement of X_1^Δ obtained by subdividing one face into three more faces. In such a situation, we find that

$$\#V = \#V_1 + 1$$

$$\#E = \#E_1 + 3$$

$$\#F = \#F_1 + 2,$$

hence the claim follows.

It is a fact that any oriented surface admits a triangulation. Hence one may compute the Euler characteristic of an oriented surface using the above procedure. In particular, I ask you to compute the Euler characteristic of the unit sphere S^2 and the torus T in the homework.

Moreover, it is a fact that we can use any polygonal representation, as defined in the previous section, to compute the Euler characteristic of a surface. I ask you to do this for projective space and the Klein bottle in the homework.

6.3.1 A surface of genus g

Let X be a path-connected, orientable surface with boundary ∂X . Define the **genus** of X to be one-half the maximum number of non-intersecting simple closed curves in X or arcs in X such that the resulting manifold obtained from cutting X along these curves is path-connected. Such simple closed curves are called **non-separating loops**.

For example, the genus of the sphere is 0, since any simple closed curve in a sphere cuts the sphere into two path-connected components. The unit disc also has genus 0, since any closed loop cuts the disc into two components, as does any arc.

The torus has genus 2, since there are two cuts along the torus which cut the torus into a manifold homeomorphic to the disc.

Moreover, a useful theorem relates the genus, Euler characteristic, and boundary components of X .

Theorem 6.6. *For a path-connected, orientable surface X with boundary ∂X , we have*

$$2g(X) = 2 - \chi(X) - |\partial X|$$

where $g(X)$ denotes the genus of X and $|\partial X|$ the number of boundary components of X .

For a non-orientable surface Y without boundary, we define the the genus by

$$\chi(Y) = 2 - g(Y).$$