Lecture # 7
The Combinatorial Approach to Knot Theory
Mike McCaffrey

1 Graphs

In this section we will define graphs in the combinatorial sense.

1.1 Definitions and Facts

Definition A *graph* is a finite set of *vertices* V, and a finite set of *edges* E. We represent each edge as an unordered pair $[v_i, v_j] \in V \times V$ of the vertices in which they connect. The vertices v_i and v_j are the *endpoints* of the edge.

E.g. An example of a graph is the following:

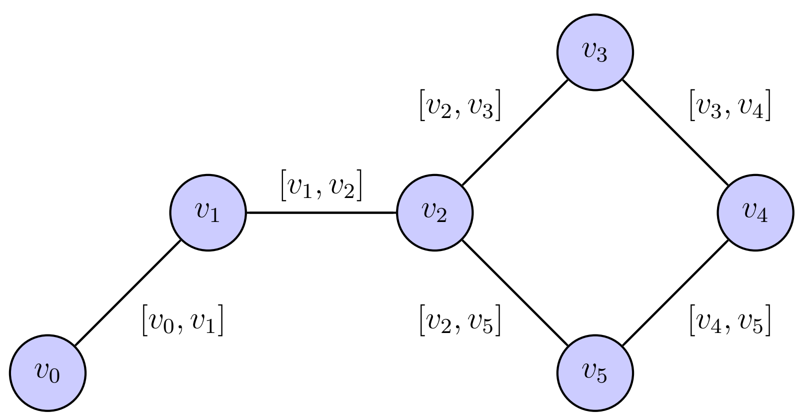


Figure 1.1

Definition An edge with the same vertex at each end is called a *loop*. When an endpoint shares two or more edges we have a *multiple edge*. A graph without loops or multiple edges is called *simple*.

E.g. An example of a multiple edge is v_3 in Figure 1.1. An example of a loop is the following:

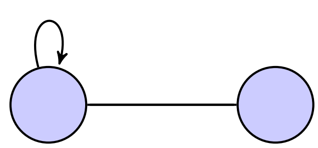


Figure 1.2

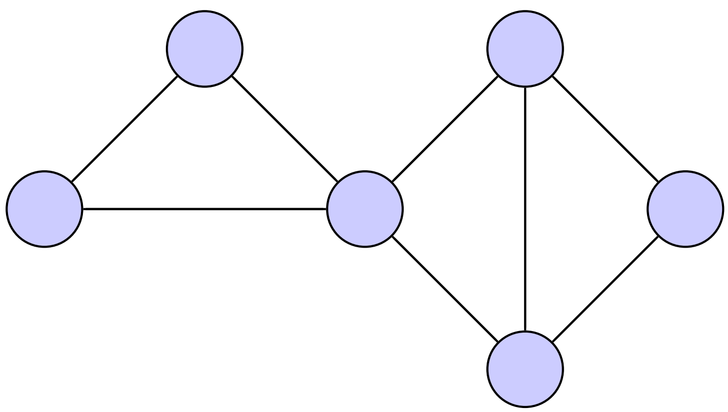
Definition The *valence* (or *degree*) of a vertex v is the number of times the vertex appears in the list of unordered pairs E (or the number of edges with endpoint v, counting loops twice).

Definition A *path* in a graph is a sequence of edges that can be written as $[v_0, v_1], [v_1, v_2], [v_2, v_3], \ldots, [v_{n-1}, v_n]$. A path is *simple* if all the v_j are distinct. A *circuit* in a graph is a path in which $v_0 = v_n$. A circuit is *simple* if all the v_j are distinct except for $v_0 = v_n$.

E.g. An example of a path would be $[v_0, v_1], [v_1, v_2], [v_2, v_3]$ in Figure 1.1.

Definition A graph is *connected* is there is a path between any two vertices. A *cut vertex* (or *articulation point*) is a vertex whose removal disconnects the graph. An *isthmus* is an edge whose removal disconnects the graph.

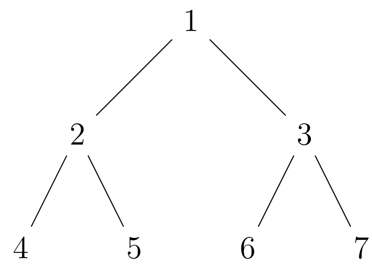
E.g. The following is an example of a connected graph with a cut vertex:



An isthmus could be illustrated by two vertices joined with one edge.

Definition A *tree* is a connected graph with no circuits. The vertices of a tree can be referred to as *nodes*. Vertices in a tree with valence $= 1$ are called *terminal nodes*, and their respective edges are called *leaves*.

E.g. A tree could be the following:



Denote a graph G with the set of vertices V and edges E as $G = (V, E)$.

Definition A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if $V' \subset V$ and $E' \subset E$.

Fact A tree has at least two terminal nodes.

Fact A tree with n nodes has $n - 1$ edges.

Proof. By induction.
Base Case: It is clear that a tree with 1 node has no edges. *Inductive Case*: Consider a tree T_{n+1} with $n + 1$ nodes. By the previous fact, there is a terminal node, so consider the sub-tree T_n created by removing the terminal node and its corresponding leaf. By our inductive assumption T_n has n nodes and $n - 1$ edges. By adding the node and edge that we removed from T_{n+1}, we deduce that T_{n+1} has n nodes and $n - 1$ edges. $\square$

In knot theory, we will mostly deal with graphs embedded into planes. We call a graph embedded into a plane a *plane* graph. A graph embedded into a plane will create regions bounded by the edges of the graph, and an unbounded region.

Definition The bounded region of a plane graph is called a *face*. A face F is *star shaped* if it contains some point x such that a straight line from x to any point on ∂F lies entirely within F. The point x is called a *star point*.

This definition allows us to derive the following theorem about plane graphs.

Fact (Fáry's Theorem) A simple plane graph can be embedded in $\mathbb{R}^2$ so that each edge is a straight line.

Note The proof of this given by Cromwell is merely a sketch, and is by induction on the vertices of a planar graph embedded in $\mathbb{R}^2$, using the concept of star points.

1.2 The Combinatorial Approach

By thinking of knots as piecewise linear we can now accomplish two things:

1. We remove the potential for wild behavior in knots.
2. We can use combinatorial methods to study them.

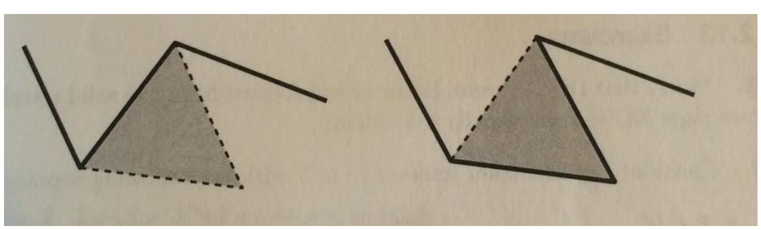
The following is an example of what thinking of knots as piecewise linear can allow us to accomplish.

Definition Let L be a polygonal link embedded in $\mathbb{R}^2$, and let Δ be a triangle such that:

- L does not meet the interior of Δ
- L meets one or two sides of $\partial\Delta$
- The vertices of L in $L \cap \Delta$ are also vertices of Δ
- The vertices of Δ in $L \cap \Delta$ are also vertices of L

We then define a Δ-move on L as a move that replaces L with $(L - (L \cap \Delta)) \cup (\partial\Delta - L)$.

E.g.



This brings us to the next definition:

Definition Two polygonal links L_1 and L_2 are said to be *combinatorically equivalent* if there is a finite sequence of Δ-moves that transforms L_1 into L_2.

Note It is clear that a Δ-move can be achieved by an ambient isotopy, and therefore we deduce that any links that are combinatorially equivalent are also ambient isotopic. The converse is also true, which leads us to the last important fact of this section:

Fact If two polygonal links are equivalent under ambient isotopy then they are also combinatorially equivalent.

Source: *Knots and Links* by Peter Cromwell