Knot Theory Notes

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3 Link Diagrams

3.1 Pictures of links

There are pros and cons of different ways of pictographically representing knots. More realistic 3d images require specifying an embedding of the knot in $\mathbb{R}^3$, a practice that often takes considerable time. Therefore many knot theorists simply draw knots as we have been formally introduced to do so in class; with a solid line, where breaks in a strand indicate the knot is passing underneath another such strand.

3.2 Projections

Since it is quite difficult to draw or otherwise produce 3d images of knots, it is common representation to represent a knot as a 2-dimensional projection (on a plane) even though knots live in 3-space.

Definition Let $L \subset \mathbb{R}^3$ be a link and let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be a projection map. A point $x \in \pi(L)$ is *regular* if $\pi^{-1}(x)$ is a single point, and is *singular* otherwise. If $|\pi^{-1}(x)| = 2$ then x is called a *double point.*

Cromwell suggests the following techniques when choosing projections:

(i) Choose projections with a *finite* number of singular points

(ii) Isolate all singular points

(iii) Avoid having two or more edges projected onto the same line

(iv) Avoid projections that render two lines tangent to one another

(v) Avoid projections that yield multiple points of high order, e.g. $|\pi^{-1}(x)| \geq 3$.

(vi) The projection should be stable, i.e. changing the angle of projection by a small amount does not drastically change the image of the link projection, nor create or destroy any singular points.

Definition The projection $\pi(L)$ of a link L is said to be *regular* if $\pi(L)$ has a finite number of singular points, and if each singular point is a transverse double point.

Theorem 3.1 *A tame link L has a regular projection.*

It is imperative to note that regular projections do not uniquely describe a link; it is indeed possible that several links may have forms of a regular projection that are the same. In the next section, we examine a solution to this.

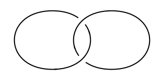
3.3 Diagrams

Determining the link type is possible if we add annotations to our regular projections which indicate which strands go over and which strands go over at intersection points. Adding this data allows us to preserve the link type, but not the actual embedding in $\mathbb{R}^3$ of the link.

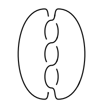
Definition A *diagram* is a regular projection of a link with relative height information at each of the double points. Conventionally, diagrams break the strand passing underneath and keep the strand continuous of the link passing overhead. These double points are referred to as *crossings*.

Theorem 3.2 *Every tame link has a diagram.*

While a diagram uniquely represents a link type, a link type cannot be uniquely represented by a diagram; i.e., one may represent the same link type by any number of different link diagrams. See the figure at right.



The trivial link



Alternative representation

Definition Let $L \subset \mathbb{R}^3$ be a link and $\pi(L)$ its projection, and D be the diagram constructed from $\pi(L)$. A *subdiagram* of D is a diagram of a sublink $L' \subset L$ using the same projection, so that $\pi(L') \subset \pi(L)$. A *component* of the diagram is a subdiagram corresponding to a 1-component sublink of L.

Definition A diagram is *connected* if its underlying projection is connected. A link with a disconnected diagram is called a *split link*.

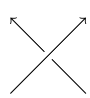
Cromwell notes that detecting diagrams with splits or diagrams that represent the unknot have an algorithmic complexity of NP hard problems.

Definition A diagram is *reducible* if the underlying projection has a cut vertex. I.e., there exists a circle in the plane of the diagram that meets the diagram at a single point which is a crossing. Cut vertices are also referred to as *nugatory* crossings. A diagram that does not have any of these crossings is said to be *irreducible* or *completely reduced*.

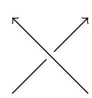
Diagrams may also be endowed with orientation;

Definition A diagram is *oriented* if each component is given an orientation or has inherited an orientation from the pre-projection link.

Definition We define a dichotomy on the crossing between two oriented pieces as follows:



Positive crossing



Negative crossing

An oriented diagram is called *positive* if each crossing is positive. (Likewise, an oriented diagram is called *negative* if each crossing is negative.

positive

Figure 3.3 from Cromwell

Definition A diagram is called *alternating* if one passes over and under alternatively as one follows each component.

Figure 3.3 from Cromwell

Definition A diagram is called *descending* if it is possible to choose a point and direction such that tracing the diagram always results in first maneuvering over the over-crossing strand at each crossing. Similarly, a diagram is called *ascending* if it is possible to choose a point and direction such that tracing the diagram always results in first maneuvering under the over-crossing stand at each crossing.

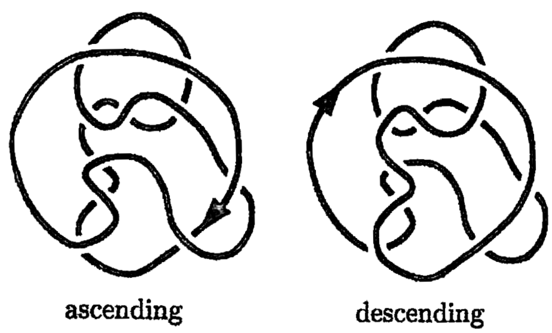


Figure 3.3 from Cromwell

Cromwell notes that an ascending or descending diagram always results in the trivial link.

Definition *Switching* a crossing is the practice of putting an under-crossing strand in the over-crossing position. (Likewise the over-crossing strand becomes the new under-crossing strand.)

Any diagram may be transformed into the diagram of the trivial link with a finite number of switchings.

Source: Cromwell, Peter R. Knots and Links. Cambridge, UK: Cambridge UP, 2004. Print.