Knot Theory Seminar
Problem Set #1
Due Wednesday, June 3

1. Show that collection τ_0 of subsets of $\mathbb{R}^n$ forms a topology on $\mathbb{R}^n$.

2. Let $S = [0,1] \subset \mathbb{R}$. Equip S with the subspace topology as a subset of $\mathbb{R}$. Show that the set $(1/2, 1]$ is open in S but that $(1/2, 1]$ is not open in $\mathbb{R}$.

3. This exercise will show that the product topology on $\mathbb{R}^2$ coincides with the standard topology τ_0 on $\mathbb{R}^2$.

(i) A **base** of a set X is a collection $\mathcal{B}$ of subsets of X satisfying the following two properties

    (a) The collection $\mathcal{B}$ covers X

    (b) For each pair of elements $U, V \in \mathcal{B}$ and each point $x \in U \cap V$, there is another basis element $W \in \mathcal{B}$ such that $x \in W \subset (U \cap V)$.

Verify that a base $\mathcal{B}$ generates a topology $\tau_\mathcal{B}$ on X whose elements consist of unions of elements of $\mathcal{B}$. More precisely, a subset U of X belongs to $\tau_\mathcal{B}$ if and only if we may write $U = \cup_{\alpha \in A} B_\alpha$ for some collection $\{B_\alpha : \alpha \in A\} \subset \mathcal{B}$.

(ii) Show that the topology $\tau_\mathcal{B}$ satisfies the following universal property: If τ is any topology on X such that $\mathcal{B} \subset \tau$, then $\tau_\mathcal{B} \subset \tau$. Conclude that $\tau_\mathcal{B}$ is the intersection of all topologies on X containing $\mathcal{B}$.

(iii) Check that if $\mathcal{B}$ and $\mathcal{B}'$ are two bases such that $\mathcal{B} \subset \mathcal{B}'$, then $\tau_\mathcal{B} \subset \tau_{\mathcal{B}'}$.

(iv) For $X = \mathbb{R}^2$, check that the collection $\mathcal{B}_0$ of ϵ-balls around points of X form a base for the standard topology τ_0. (Hint: Take an open set in the standard topology and show that it can be written as a union of ϵ-balls.)

(v) For $X = \mathbb{R}^2$, show that the collection $\mathcal{B}'$ of subsets of the form $(a, b) \times (c, d)$ form a basis for the product topology on $\mathbb{R}^2$.

(vi) For $X = \mathbb{R}^2$, verify that $\mathcal{B}_0 \subset \tau_{\mathcal{B}'}$ and $\mathcal{B}' \subset \tau_{\mathcal{B}_0}$, and conclude that the standard topology coincides with the product topology.

4. Prove directly that any constant map $f : X \to Y$ given by $f(x) = c$ for some fixed $c \in Y$ is continuous.

5. Suppose that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ satisfies the following property

(*) For each point $x \in \mathbb{R}^n$ and each $\epsilon > 0$, there is a $\delta > 0$ so that $f(B_\delta(x)) \subset B_\epsilon(f(x))$.

Show that f is a continuous map of topological spaces. Conclude that functions which are known to be continuous from Calculus are also continuous in this new topological sense.

6. Check that the composition of two continuous maps is continuous.

7. Let X, X_1, X_2 be topological spaces, and for a map $f : X \to X_1 \times X_2$, write $f(x) = (f_1(x), f_2(x))$ for some $f_i : X \to X_i$. Show that f is continuous if and only if both f_1 and f_2 are.

8. On the other hand, let $f : X_1 \times X_2 \to X$ be a map between topological spaces. For a fixed $x_1 \in X_1$, we may define $f_{x_1} : X_2 \to X$ by $f_{x_1}(x_2) = f(x_1, x_2)$. We may similarly define $f_{x_2} : X_1 \to X$ for a fixed $x_2 \in X_2$. If both f_{x_1} and f_{x_2} are continuous for each $x_1 \in X_1$ and each $x_2 \in X_2$, is it true that f is continuous? Proof or counterexample.

9. Show that the product topological space $X_1 \times X_2$ satisfies the following universal property: there are continuous maps $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$ such that for each topological space Y and each

pair of continuous maps $f_1 : Y \to X_1$ and $f_2 : Y \to X_2$, there is a unique continuous map $f : Y \to X_1 \times X_2$ such that the following diagram commutes

$$
\begin{array}{ccc}
 & Y & \\
 f_1 \swarrow \quad & \downarrow f & \quad \searrow f_2 \\
X_1 \xleftarrow{\;\pi_1\;} & X_1 \times X_2 & \xrightarrow{\;\pi_2\;} X_2
\end{array}
$$

10. A space X is called **path-connected** if for any pair of points $x, y \in X$, there is a continuous map $\phi : [0, 1] \to X$ such that $\phi(0) = x$ and $\phi(1) = y$.

(i) Show that the unit circle S^1 is path connected.

(ii) A subset S of $\mathbb{R}^n$ is called **convex** if for any pair of points $x, y \in S$, the line segment $\{tx + (1 - t)y : t \in [0, 1]\}$ belongs to S. Show that a convex subset is path connected.

(iii) On the other hand, find a path connected subset that is not convex.

(iv) Show that the unit circle S^1 is not homeomorphic to the unit interval $(0, 1)$. (Hint: What happens when you remove a point of S^1 and what happens when you remove a point of $(0, 1)$?)

11. Let Y denote the quotient space of $X = [0, 1]$ with the equivalence relation described in the notes, and let $f : [0, 1] \to S^1 \subset \mathbb{C}$ be the map described by $f(t) = e^{2\pi i t}$. Show that f induces a well-defined map $\phi : Y \to S^1$ which is a homeomorphism.

12. Let X be a topological space with equivalence relation $\sim$, and let $\pi : X \to Y$ denote the natural projection onto the set of equivalence classes Y, equipped with the quotient topology. Show that the quotient map $\pi : X \to Y$ satisfies the following universal property: If Z is another topological space, and $f : Y \to Z$ is any map, then f is continuous if and only if $f \circ \pi$ is continuous.

13. Let $B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$ denote the unit ball in $\mathbb{R}^n$. Let $f : B_1(0) \to B_1(0)$ denote the identity map and $g : B_1(0) \to B_1(0)$ by the constant map $g(x) = 0$. Find a homotopy from f to g.

14. Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$. Let $f : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{R}^{n+1} \setminus 0$ denote the identity map on on the complement of the origin in $\mathbb{R}^{n+1}$. Let $g : \mathbb{R}^{n+1} \setminus 0 \to S^n \subset \mathbb{R}^{n+1} \setminus 0$ denote the map described by

$$g(x) = \frac{x}{|x|}.$$

Show that f is homotopic to g.

15. Exercise 1.1 of Cromwell.