Knot Theory Seminar
Problem Set #3
Due Wednesday, June 10

1. (With motivation from Pedro) Let X, Y be topological spaces with bases $\mathcal{B}$ and $\mathcal{B}'$ respectiely. Let $f : X \to Y$ be a map (not necessarily continuous).

(i) Show that if $f^{-1}(B) \in \mathcal{B}$ for each $B \in \mathcal{B}'$, then f is continuous.

(ii) Show that if $f^{-1}(B)$ is open for each $B \in \mathcal{B}'$, then f is continuous.

(iii) Show that the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ does not satisfy the hypothesis of (i), yet satisfies the hypothesis of (ii) and hence is continuous. Here, use the basis of $\mathbb{R}$ consisting of open intervals.

(iv) Is the converse of (i) or (ii) true? Proof or counterexample.

2. Recall that a space X is **compact** if every open cover admits a finite subcover. A subset C of a space X is called **closed** if $X \setminus C$ is open.

(i) Show that a closed subset of a compact space is compact.

(ii) Show that the continuous image of a compact space is compact. (That is, show that if $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.)

(iii) Show that a map $f : X \to Y$ between topological spaces is continuous if and only if $f^{-1}(C)$ is closed for each closed set $C \subset Y$.

3. Recall that a space X is **Hausdorff** if for each pair of distinct points $x, y \in X$, there are disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.

(i) Show that $\mathbb{R}^n$ is Hausdorff (with respect to the standard topology).

(ii) Define a topology τ^* on $\mathbb{R}$ by declaring $U \in \tau^*$ if and only if one of the two conditions is satisfied

- $\mathbb{R} \setminus U$ consists of a finite number (possibly zero) of points or
- $U = \varnothing$.

Show that τ^* is a topology on $\mathbb{R}$ (called the **co-finite topology**).

(iii) Show that $\mathbb{R}$ is not Hausdorff with respect to the topology τ^*.

(iv) More generally, let X be a set and equip X with the co-finite topology (so that the open sets are the empty set and those sets that have finite complement). Is X Hausdorff? Proof or counterexample.

4. Let U be a subset of a topological space X. Show that if for each point $x \in U$ there is an open set V such that $x \in V \subset U$, then U is open. Is the converse true? Proof or counterexample.

5. Show that a compact subset of a Hausdorff space is closed. (Hint: Let C be compact in a Hausdorff X. Fix a point $x \in X \setminus C$. For each $y \in C$, there are disjoint open $U_y \ni x$ and $V_y \ni y$. There are a finite number of points $y_1, \ldots, y_n \in C$ such that $C \subset \cup_{i=1}^n V_{y_i}$. Then $U = \cap_{i=1}^n U_{y_i}$ is an open neighborhood of x disjoint from C, so we are done.)

6. On the other hand, show that any subset of $\mathbb{R}$ is compact with respect to the topology τ^* of Problem 3. Why does this not contradict Problem 5?

7. Let $f : X \to Y$ be a continuous bijection from a compact space X to a Hausdorff space Y. Show that the inverse g of f is continuous. (Hint: Use 2(iii) by using 2(i), 2(ii), and 5 in that order.)

8. If X is a compact space with equivalence relation $\sim$, show that the quotient space $Y = X/\sim$ is compact.

9. Let $X = [0, 1]$ withe equivalence relation $\sim$ that identifies 0 and 1, and let Y be the resulting quotient space. Let $\phi : Y \to S^1$ denote the well-defined map induced by that map $f : [0, 1] \to S^1$ given by $f(t) = e^{2\pi it}$. In class, we showed that ϕ is continuous. We also showed that ϕ is bijective, and hence admits an inverse $\psi : S^1 \to Y$. Show that ψ is continuous.

10. Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a continuous function. Show that the graph of f

$$\Gamma(f) = \{(x, f(x)) \in \mathbb{R}^{n+1} : x \in U\}$$

is an n-dimensional manifold.

11. Show that $\mathbb{R}^n$ with the standard topology is second-countable.

12. Show that S^n is an n-dimensional manifold. (Hint: Cover S^n by the following $2n+2$ coordinate charts: For $i = 1, \dots, n+1$, let U_i^+ denote the subsets of S^n given by

$$U_i^+ = \{(x_1, \dots, x_{n+1} \in S^n : x_i > 0\},$$

and define U_i^- similarly. Show that the collection of $U_i^\pm$'s covers S^n and that each $U_i^\pm$ is the graph of a continuous one-to-one function $f : B_1(0) \to \mathbb{R}$ where $B_1(0)$ denotes the open unit ball of radius 1 centered at the origin in $\mathbb{R}^n$.)

13. Use polygonal representations or triangulations to compute the Euler characteristics and genera of the

(i) Unit sphere S^2

(ii) Torus T

(iii) Klein bottle K

(iv) Projective space.

14. For orientable surfaces X, Y, find a formula relating $\chi(X\#Y)$, $\chi(X)$, and $\chi(Y)$ and prove your result.

15. If M and N are manifolds of dimension m and n respectively, show that $M \times N$ enjoys the structure of a $(m+n)$-dimensional manifold.