Knot Theory Seminar
Problem Set #4
Due Wednesday, June 17

**1.** As a set, let $X = \mathbb{R} \sqcup \{0^*\}$, where $0^*$ is some element. Define the open sets of $X$ to be those of the form

- $U$ for $U$ open in $\mathbb{R}$
- $U \cup \{0^*\}$ for $U$ open in $\mathbb{R}$ containing 0.
- $(U \setminus \{0\}) \cup \{0^*\}$ for $U$ open in $\mathbb{R}$ and containing 0.

(i) Check that this defines a topology on $X$.

(ii) Check that $X$ is locally Euclidean of dimension 1.

(iii) Check that $X$ is not Hausdorff.

This space $X$ is called the line with the origin doubled. This exercise shows that, contrary to my belief during lecture, locally Euclidean does not imply Hausdorff.

**2.** Show that a space $X$ is Hausdorff if and only if the diagonal $\Delta = \{(x, y) \in X \times X : x = y\}$ is closed in $X \times X$.

**3.** More generally, if $X$ is a space with an equivalence relation $\sim$ and the projection $\pi : X \to X/\sim$ is an open map (meaning that $\pi(U)$ is open for each open $U \subset X$), show that the quotient space $X/\sim$ is Hausdorff if and only if $\sim$ is closed in $X \times X$.

**4.** Show that the unit circle $S^1$ is compact. (Hint: Use the previous problem set.)

**5.** Show that a knot (even a wild one) is compact.

**6.** The goal of this problem is to provide further justification for why polygonal knots are locally flat at the vertex points.

(i) For a point $t_1 \in (0, 1)$, show that there is a homeomorphism $h : [0, 1] \to [0, 1]$ such that $h(t_1) = \frac{1}{2}$.

(ii) Let $r_0$ denote the radius of the unit disc $D_1$ in $\mathbb{R}^2$ along the positive $x$-axis, and let $r_1$ denote a radius such that the sector cut out by $r_0$ and $r_1$ has angle $2\pi t_1$. Let $R$ denote the arc given by the union of the two radii $r_0$ and $r_1$. Show that there is a homeomorphism of the open unit disc $H : D_1 \to D_1$ such that $H(R)$ is the intersection $\{x\text{-axis}\} \cap D_1$. (Hint: Use polar coordinates and (i).)

(iii) Show that for a vertex $v$ in a polygonal knot $K$, there is an $\epsilon > 0$ and a homeomorphism $G : B_\epsilon(v) \to B_1(0)$ from $B_\epsilon(v)$ to the unit ball in $\mathbb{R}^3$ such that $G(K \cap B_\epsilon(v))$ is a diameter of the ball $B_1(0)$. (Hint: Use spherical coordinates and (ii)).

**7.** Cromwell 3.10.4.