Knot Theory Seminar
Problem Set #5
Due Tuesday, June 23

Note! For sets X, Y, I write $X \subset Y$ to mean that X is a subset of Y. If I want to indicate that X is a *proper* subset of Y, then I write $X \subsetneq Y$.

1. For a subset S of a topological space X, we say that a point $x \in S$ is an **interior point of** S if there is an open subset U of X such that $x \in U \subset S$. We let $\mathrm{Int}(S)$ denote the set of interior points of S.

(i) Show that

$$\mathrm{Int}(S) = \bigcup_{\substack{U \text{ open} \\ U \subset S}} U$$

and deduce that $\mathrm{Int}(S)$ is open.

(ii) Proof or counterexample: S is open if and only if $S = \mathrm{Int}(S)$.

(iii) Proof or counterexample: if $S_1 \subset S_2$, then $\mathrm{Int}(S_1) \subset \mathrm{Int}(S_2)$.

(iv) Proof or counterexample: $\mathrm{Int}(S_1 \cap S_2) = \mathrm{Int}(S_1) \cap \mathrm{Int}(S_2)$.

(v) Proof or counterexample: $\mathrm{Int}(S_1 \cup S_2) = \mathrm{Int}(S_1) \cup \mathrm{Int}(S_2)$.

2. For a subset S of a topological space X, we let $\mathrm{Cl}(S)$ denote the closure of S:

$$\mathrm{Cl}(S) = \bigcap_{\substack{C \text{ closed} \\ C \supset S}} C.$$

(i) Show that the intersection of closed sets is closed and deduce that $\mathrm{Cl}(S)$ is the "smallest closed set containing S."

(ii) Proof or counterexample: S is closed if and only if $\mathrm{Cl}(S) = S$.

(iii) Proof or counterexample: if $S_1 \subset S_2$, then $\mathrm{Cl}(S_1) \subset \mathrm{Cl}(S_2)$.

(iv) Proof or counterexample: $\mathrm{Cl}(\mathrm{Cl}(S)) = \mathrm{Cl}(S)$.

(v) Proof or counterexample: $\mathrm{Cl}(S_1 \cap S_2) \subset \mathrm{Cl}(S_1) \cap \mathrm{Cl}(S_2)$.

(vi) Proof or counterexample: $\mathrm{Cl}(S_1 \cap S_2) = \mathrm{Cl}(S_1) \cap \mathrm{Cl}(S_2)$.

(vii) Proof or counterexample: $\mathrm{Cl}(S_1 \cup S_2) = \mathrm{Cl}(S_1) \cup \mathrm{Cl}(S_2)$.

(viii) Proof or counterexample: $\mathrm{Cl}(S) = X \setminus (\mathrm{Int}(X \setminus S))$.

3. For a subset S of a topological space X, we say that $x \in X$ is a **limit point of** S if each open neighborhood U of x intersects S in at least one point other than x itself.

(i) Show that S is closed if and only if it contains all its limit points.

(ii) If $L(S)$ denotes the set of limit points of S, deduce that $\mathrm{Cl}(S) = S \cup L(S)$.

(iii) Find $L(B_1(0))$ where $B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$.

(iv) Find $L((a, b))$ and $L([a, b])$.

4. Define the boundary of $S \subset X$ to be $\partial S = \mathrm{Cl}(S) \setminus \mathrm{Int}(S)$.

(i) Show that $\partial(S_1 \cap S_2) \subset (\partial S_1 \cap \mathrm{Cl}(S_2)) \cup (\mathrm{Cl}(S_1) \cap \partial S_2)$. But show that the reverse inclusion is in general not true.

(ii) Show that ∂ satisfies the Leibniz rule $\partial(S_1 \cap S_2) = (\partial S_1 \cap S_2) \cup (S_1 \cap \partial S_2)$ if both S_1 and S_2 are closed.

(iii) Proof or counterexample: ∂S is closed.

(iv) Find $\partial(B_1(0))$ where $B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$.

(v) Find $\partial(S^n)$ where $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$.

(vi) Find $\partial(D)$ where $D = \{x \in \mathbb{R}^n : |x| \leq 1\}$.

(vii) Find $\partial((2,3))$ and $\partial([2,3])$.

(viii) Find $\partial(\mathbb{Q})$ where $\mathbb{Q} \subset \mathbb{R}$ denotes the subset of rational numbers.

(ix) Proof or counterexample: $\partial\partial S = \partial S$.

(x) Proof or counterexample: $\partial\partial\partial S = \partial\partial S$.

5. Let X be a topological space. Consider the following conditions on X

(a) each point $x \in X$ admits an open neighborhood $U \ni x$ and a continuous map $\phi : U \to \mathbb{R}^n$ taking U homeomorphically onto the open unit ball $B_1(0) \subset \mathbb{R}^n$

(b) each point $x \in X$ admits an open neighborhood $U \ni x$ and a continuous map $\phi : U \to \mathbb{R}^n$ taking U homeomorphically onto an open subset $\phi(U) \subset \mathbb{R}^n$

(c) each point $x \in X$ admits an open neighborhood $U \ni x$ and a continuous map $\phi : U \to \mathbb{R}^n_{\geq 0} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$ taking U homeomorphically onto an open subset $\phi(U) \subset \mathbb{R}^n_{\geq 0}$.

(i) Construct a homeomorphism from $\mathbb{R}^n_{>0} = \{(x_1, \ldots, x_n) : x_1 > 0\}$ onto $\mathbb{R}^n$.

(ii) Show that $(a) \implies (b) \implies (c)$.

(iii) Show that $(b) \implies (a)$ (and hence $(a) \iff (b)$)

(iv) Proof or counterexample: $(c) \implies (b)$.

6. For $m \geq n$, and for a map $\sigma : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$, let $f_\sigma : \mathbb{R}^m \to \mathbb{R}^m$ be defined by

$$f_\sigma(x_1, \ldots, x_m) = (x_{\sigma(1)}, \ldots, x_{\sigma(m)}).$$

Find necessary and sufficient conditions on m, n, and σ so that f_σ is a homeomorphism.

7. Let $\{U_\alpha\}$ be an open cover of a topological space X.

(i) Proof or counterexample: a subset V is open in X if and only if each $V \cap U_\alpha$ is open in U_α.

(ii) Proof or counterexample: a subset C is closed in X if and only if each $C \cap U_\alpha$ is closed in U_α.

8. For a space X satisfying condition (c) above, let δX denote those points $x \in X$ which admit an open neighborhood $U \ni x$ and a continuous map $\phi : U \to \mathbb{R}^n_{\geq 0}$ taking U homeomorphically onto $\phi(U)$ such that $\phi(x) \in \{x_1 = 0\} \subset \mathbb{R}^n$. Proof or counterexample: If $X \subset \mathbb{R}^n$, then $\partial X = \delta X$.

9. Cromwell 3.10.6

10. Cromwell 3.10.9

11. Cromwell 3.10.14 (Note that a **chessboard colouring** is defined in 3.10.13.)

12. Cromwell 4.11.2