Knot Theory Seminar
Problem Set #6 Selected Solutions

I've written down some solutions. I would suggest taking a look at

- Problem 1 (iv), (vi), (viii)
- Problem 2 (vi)
- Problem 3 (iii)
- Problem 4
- Problem 6

These seemed to give the most trouble.

I also have some remarks about writing your proofs.

- If you are going to use the symbol $\implies$, which I suggest you don't, then each appearance of $\implies$ must supported with an appeal to a definition, proposition, theorem, etc. Otherwise, the reader has no idea why the next statement follows. I suggest never to use this symbol, instead writing out the correct logical sentences that convey exactly what you mean to say, for the increased satisfaction of your reader (and yourself when you try to review).
- I would suggest to write out the statement of the problem. This is to the benefit of the grader and yourself, when you wish to review.
- When proving an if-and-only-if statement, I think it is most practical to simply prove both directions separately. It is an exercise in and of itself when reading a chain of if-and-only-if statements to verify that all directions are indeed true. Moreover, I despise the symbol $\implies$ in written work (see bullet 1), so I especially despise the symbol $\iff$. Although certain situations permit you to "reverse all the arrows," these situations are less frequent the further you travel in math, and it is perhaps better practice to get used to the traditional method of proof.
- I would suggest writing in complete sentences. Your reader will be extremely happy with you when you do this. Moreover, when you speak about math, you wish to do so in complete sentences, so why not when writing about it too? Moreover, when you write math at any other level, it is expected that it will be written in complete sentences.
- When completing a proof by contradiction, it is often good practice to explicitly state your goal in the proof: "Suppose, toward a contradiction, that X holds..." Then alert your reader when the contradiction has been reached: "..., which is a contradiction to the assumption that Y holds..." Then get rid of your false assumption: "Thus our assumption about X was false, and hence X does not hold."
- Though the symbols $\exists$ and $\forall$ may be fun to write and good for note-taking, they don't really have a place in written mathematics, as they are difficult to read (unless used well). Remember: your reader is a human being—not a computer—and will appreciate reading written work in his/her native language, not some mix of english words and symbols that are sometimes difficult to parse.
- Finally, organization and clarity can improve your reputation with the reader. Hence, I would suggest writing neatly and legibly, organizing your work neatly between the margins, using new paragraphs to signify transitions in proofs, writing your solutions in order, finishing your proofs with a $\square$, and leaving enough space for comments.

1. I feel like I didn't explain my response to Andrew's question properly, so here is an exercise which fills in the details better. Recall that a map $f : X \to Y$ is open if and only if $f(U)$ is open for each open $U \subset X$. Andrew asked whether there were examples of projection maps $\pi : X \to X/\sim$ which are not open. The answer is yes, and here is one line of reasoning that I tried to outline on the board.

(i) For a topological space X, a set Y, and a surjective mapping $f : X \to Y$, show that the set $\{V \subset Y : f^{-1}(V) \text{ is open in } X\}$ is a topology on Y called the **quotient topology**.

Proof. Because f is a map, we have $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(Y) = X$. This shows that $\varnothing$ and Y belong to the quotient topology.

Because the preimage under f commutes with taking unions or taking intersections, we find that unions of open sets are open and finite intersections of open sets are open. □

(ii) When Y is equipped with the quotient topology, show that f is continuous.

Proof. If $V \subset Y$ is open, then by definition of the quotient topology, $f^{-1}(V)$ is open. □

(iii) Define an equivalence relation $\sim$ on X by $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$, and let $X/\sim$ denote the quotient space. Show that f induces a well-defined map $g : X/\sim \to Y$ described by $g([x]) = f(x)$.

Proof. If $x_1, x_2 \in X$ satisfy $x_1 \sim x_2$, then by definition $f(x_1) = f(x_2)$, and hence $g([x_1]) = f(x_1) = f(x_2) = g([x_2])$. □

(iv) Show that g is continuous.

Proof. Let V be an open subset of Y.
By definition of the quotient topology, $f^{-1}(V)$ is open in X.
Because $f = g \circ \pi$, we have $f^{-1}(V) = \pi^{-1}(g^{-1}(V))$.
By definition, a subset W of $X/\sim$ is open in $X/\sim$ if and only if $\pi^{-1}(W)$ is open in X.
Hence $g^{-1}(V)$ is open in $X/\sim$. □

(v) Show that g is bijective and hence admits an inverse $g^{-1} : Y \to X/\sim$.

Proof. We first show that g is injective. Suppose $[x_1], [x_2] \in X/\sim$ satisfy $g([x_1]) = g([x_2])$. Then $f(x_1) = f(x_2)$ meaning that $x_1 \sim x_2$ or equivalently $[x_1] = [x_2]$.

We next show that g is surjective. Let $y \in Y$ be arbitrary. Because f is surjective, there is an $x \in X$ such that $f(x) = y$. Then $g([x]) = y$. □

(vi) Show that the inverse of g is continuous, and hence g is a homeomorphism.

Proof. Let V be an open subset of $X/\sim$.
Then by definition of the quotient space, $\pi^{-1}(V)$ is open in X.
Because $\pi = g^{-1} \circ f$, we have $\pi^{-1}(V) = f^{-1}((g^{-1})^{-1}(V))$.
A subset U of Y is open if and only if $f^{-1}(U)$ is open in X.
Hence $(g^{-1})^{-1}(V)$ is open in Y. □

(vii) Show that f is open if and only if π is.

Proof. A homeomorphism is an open map, so in particular g and g^{-1} are open.
The composition of open maps is open.
Hence if f is open, then $\pi = g^{-1} \circ f$ is open.
Conversely, if π is open, then $f = g \circ \pi$ is open. □

(viii) As sets, let $X = Y = [0, 1]$. Equip X with the standard topology. Let $f : X \to Y$ be defined by

$$f(x) = \begin{cases} 0 & 0 \leq x \leq 1/2 \\ 2x - 1 & 1/2 \leq x \leq 1. \end{cases}$$

Equip Y with the quotient topology induced by f. Show that f is continuous but not open. Conclude that the corresponding projection map $\pi : X \to X/\sim$ is not open.

Proof. The map f is continuous by (ii).
Let U be the open subset of X given by $U = [0, 1/4)$.
Then $f(U) = \{0\}$.
Note that $f^{-1}(\{0\}) = [0, 1/2]$, which is not open in X.
By definition of the quotient topology, $f(U)$ is not open in Y.
Hence f is not open.
By part (vi), π is not open. $\square$

2. Recall that an **abelian semigroup** is a set S together with a binary operation $* : S \times S \to S$ such that $s_1 * s_2 = s_2 * s_1$ for each $s_1, s_2 \in S$. We say that S has a **unit** if there is an element $e \in S$ such that $e * s = s$ for each $s \in S$. Theorem 4.6.2 says that $\mathbb{K}$ is an abelian semigroup with unit given by the unknot.

(i) If S has a unit, show that it is unique. (That is, if e_1, e_2 are two units for S, show that $e_1 = e_2$.)

Proof. If e_1 and e_2 are units, then

$$\begin{aligned} e_1 &= e_2 * e_1 && e_2 \text{ a unit} \\ &= e_1 * e_2 && * \text{ commutative} \\ &= e_2 && e_1 \text{ a unit.} \end{aligned}$$

$\square$

(ii) We say that an element $s \in S$ **divides** another element $r \in S$, if there is an element $t \in S$ such that $s * t = r$. Show that the unknot divides every knot.

Proof. A knot K is locally flat, so there is a factorizing sphere S whose interior U is such that $(U, U \cap K)$ is homeomorphic to the unit ball in $\mathbb{R}^3$ with a diameter. Let α be an arc in S connecting the two points of $S \cap K$. Then $(U \cap K) \cup \alpha$ bounds a disc, and hence is a factor of K ambient isotopic to the unknot. $\square$

(iii) We say that an element s is **prime in** S if whenever s divides a product $a * b$, either s divides a or s divides b. Show that if K_P is a prime knot, then K_P is a prime element of $\mathbb{K}$.

Proof. Say that K_P divides $K_A \# K_B$.
By Theorem 4.5.2, K_P either divides K_A or K_B. $\square$

(iv) We say that a non-unit $s \in S$ is **irreducible** if whenever $s = s_1 * s_2$ for some $s_1, s_2 \in S$, either $s_1 = e$ or $s_2 = e$. Show that every prime number is irreducible in $(\mathbb{N}_{>0}, \cdot)$.

Proof. A prime number p satisfies the property that p and 1 are its only factors.
If p is equal to a product ab, then either $a = 1$ or $b = 1$.
Since 1 is the unit of $(\mathbb{N}_{>0}, \cdot)$, it follows that p is irreducible. $\square$

(v) Show that every prime knot is irreducible in $\mathbb{K}$.

Proof. Let K be a prime knot.
Then by definition, we may not write $K = K_A \# K_B$ unless either K_A or K_B is trivial. $\square$

(vi) We say that an abelian semigroup with unit S has unique factorization if for each element $s \in S$ there are irreducible elements $s_1, \ldots, s_n \in S$ such that

$$s = e * s_1 * \cdots * s_n$$

and this representation is unique in the sense that if

$$s = e * t_1 \cdots * t_m$$

for some $t_1, \ldots, t_m \in S$, then $m = n$ and there is a bijection $\phi : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that $t_{\phi(i)} = s_i$ for each i. Why does $\mathbb{K}$ have unique factorization?

Proof. Let K be a knot. By Theorem 4.5.1, there are a finite number of knots $K_1, \ldots, K_n$ such that

$$K = K_1 \# \cdots \# K_n.$$

Moreover, we may assume that each K_i is prime, or else we could reduce further. Hence, by part (v), we may assume that each K_i is irreducible. Then we have the existence we need:

$$K = E \# K_1 \# \cdots \# K_n$$

Now, suppose that we may also write K as $L_1 \# \cdots \# L_m$ for irreducible L_i. We prove by induction on n that $m = n$ and there is a permutation σ of $\{1, \ldots, n\}$ such that $K_i = L_{\sigma(i)}$ for all i.

Suppose that $n = 1$. Then $K_1 = L_1 \# \cdots \# L_m$. Because K_1 is irreducible, either L_1 or $L_2 \# \cdots \# L_m$ is the unknot. But L_1 cannot be the unknot, as L_1 is not a unit. It follows that $L_2 \# \cdots \# L_m$ is the unknot. This is impossible unless this connected sum is empty. It follows that $m = 1$, and $L_1 = K_1$.

Suppose now that the claim holds for n, and consider the case for $n + 1$. In this case, we have

$$K_1 \# \cdots \# K_{n+1} = L_1 \# \cdots \# L_m.$$

By Theorem 4.5.2, K_{n+1} divides some L_j. Without loss of generality, we may suppose that $L_j = L_m$. Since L_m is irreducible, we must have that $K_{n+1} = L_m$. By Theorem 4.5.3, we can cancel K_{n+1} and L_m from the above equation, to find that

$$K_1 \# \cdots \# K_n = L_1 \# \cdots \# L_{m-1}.$$

By the induction hypothesis, we have $n = m - 1$ and there is a permutation of $\{1, \ldots, n\}$ such that $K_i = L_{\sigma(i)}$ for each i. It follows that $n + 1 = m$ and there is a permutation σ' of $\{1, \ldots, n + 1\}$ such that $K_i = L_{\sigma'(i)}$ for each i. This completes the inductive step and the proof. □

3. This exercise supplements the proof of Lemma 4.7.1 in Cromwell. Let $v = (v_1, v_2, v_3)$ be a vector in $\mathbb{R}^3$ such that $v_3 \neq 0$. If $H_+ = \{x_3 > 0\} \subset \mathbb{R}^3$ and $H_- = \{x_3 < 0\} \subset \mathbb{R}^3$, then either $v \in H_+$ or $v \in H_-$.

(i) If $v \in H_\pm$, show that there is a unique linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying $L(e_1) = e_1, L(e_2) = e_2$, and $L(v) = \pm e_3$.

Proof. In linear algebra, the following statement is usually proved: If $v_1, \ldots, v_n$ is a basis for a vector space V and $w_1, \ldots, w_n$ are vectors in a vector space W, then there is a unique linear map $L : V \to W$ satisfying $L(v_i) = w_i$ for each i. Hence, for our purposes, it suffices to show that $\mathcal{B} = \{e_1, e_2, v\}$ is a basis for $\mathbb{R}^3$. But this is clear: Indeed the 3-by-3 matrix whose columns are the vectors of $\mathcal{B}$ has nonzero determinant, and hence is invertible. □

(ii) Conclude that L is the identity on the x_1x_2-plane, and sends v to a vector perpendicular to this plane.

Proof. If w is in the x_1x_2-plane, then there are $w_1, w_2 \in \mathbb{R}$ such that $w = w_1e_1 + w_2e_2$. Since L is linear, we have

$$L(w) = w_1L(e_1) + w_2L(e_2) = w_1e_1 + w_2e_2 = w.$$

This shows that L is the identity on the x_1x_2-plane.

The vectors $\pm e_3$ are perpendicular to the x_1x_2-plane, so the second part of the claim follows. □

(iii) Show that L is orientation preserving.

Proof. By definition, L is orientation preserving if and only if its derivative has positive determinant. Since L is linear, its derivative is equal to itself. The determinant of L is defined to be the determinant of the matrix of L in any basis for $\mathbb{R}^3$. In terms of the standard basis, the determinant is easily computed to be $\frac{1}{|v_3|}$, which is positive. □

(iv) Show that there is an isotopy from the identity map to L.

Proof. Let $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ denote the map defined by

$$H(x,t) = H_t(x) = (1-t)x + tL(x).$$

Then $H(x,t)$ is a continuous map in t and x and satisfies $H_0(x) = x$ and $H_1(x) = L(x)$. Moreover, for a fixed $t \in [0,1]$, the determinant of the linear map H_t is $\det H_t = (1-t) + t/|v_3|$, which is positive, and hence H_t is an isomorphism, and thus a homeomorphism. □

4. Cromwell 4.11.2. Show that companionship is reflexive and transitive.

Proof. We first show that companionship is reflexive. Let C be a knot, with tubular neighborhood V. Let P be the unknot with tubular neighborhood W. Let h be a homeomorphism from W onto V. Then C is a companion of $h(P)$. But $h(P)$ is ambient isotopic to the core C of V. Hence C is a companion of itself.

We next show that companionship is transitive. Say that C_1 is a companion of C_2 and C_2 is a companion of C_3. This means that we can find a pattern P_1 in the standard solid torus W, a tubular neighborhood V_1 of C_1, and a homeomorphism $h_1 : W \to V_1$ such that $h_1(P_1) = C_2$. We can also find a pattern P_2 in the standard solid torus W, a tubular neighborhood V_2 of C_2, and a homeomorphism $h_2 : W \to V_2$ such that $h_2(P_2) = C_3$. We may assume that C_2 is in the interior of V_1 and C_3 in the interior of V_2. Hence, by shrinking the tubular neighborhood V_2 if necessary, we may assume that $V_2 \subset V_1$.

Let P be the pattern in W defined by $P = h_1^{-1}(h_2(P_2))$. Then h_1 is a homeomorphism from W to V_1 such that $h_1(P) = h_2(P_2) = C_3$. Hence by definition C_1 is a companion of C_3. □

5. Cromwell 4.11.7.

Proof. Suppose that K has unknotting number 1. Say we may write $K = K_1 \# K_2$. Because we are assuming the unknotting conjecture to be true, we have $1 = \mu(K) = \mu(K_1) + \mu(K_2)$. Since $\mu(K_i) \geq 0$ for each i, without loss of generality we have $\mu(K_1) = 0$ and $\mu(K_2) = 1$. Any knot with unknotting number 0 is the trivial link, so K_1 is trivial. It follows that K is prime by definition. □

6. Cromwell 4.11.8. Let S be a factorizing sphere for a knot K, and let $\lambda \subset S$ be a single loop that is disjoint from K. Show that the linking number $\operatorname{lk}(\lambda, K)$ is 0 or ± 1.

Proof. The intersection $S \cap K$ consists of two points. The loop λ separates S into two discs Δ_1 and Δ_2. Hence, each point lies in one of the discs, not both. Without loss of generality are two cases to consider (i) Δ_1 has two points and (ii) Δ_1 has one point.

(i) If Δ_1 has two points, then we may perform surgery by cutting along λ and gluing discs to the resulting pieces so that one is a sphere S' containing K in its interior. Hence now S' is a splitting sphere for the link $K \cup \lambda$, which implies that $K \cup \lambda$ has linking number 0.

(ii) If Δ_1 has one point, then I claim there is a diagram for $K \cup \lambda$ which has exactly two crossings: Indeed, we may shrink λ so that it is a very small loop around one point of $K \cap S$—and hence around one strand of K—from which it is clear that the number of crossings in a suitable diagram will be two. One crossing must be an over-crossing and the other an under-crossing (or else Δ_1 would not have any points of $K \cap S$ in it). The crossings will have the same orientation, and hence $\operatorname{lk}(K, \lambda) = \pm 1$.

This completes the proof. $\square$

7. Cromwell 4.11.9.