Knot Theory Seminar
Problem Set #6
Due Monday, June 29

1. I feel like I didn't explain my response to Andrew's question properly, so here is an exercise which fills in the details better. Recall that a map $f : X \to Y$ is open if and only if $f(U)$ is open for each open $U \subset X$. Andrew asked whether there were examples of projection maps $\pi : X \to X/\sim$ which are not open. The answer is yes, and here is one line of reasoning that I tried to outline on the board.

(i) For a topological space X, a set Y, and a surjective mapping $f : X \to Y$, show that the set $\{V \subset Y : f^{-1}(V) \text{ is open in } X\}$ is a topology on Y called the **quotient topology**.

(ii) When Y is equipped with the quotient topology, show that f is continuous.

(iii) Define an equivalence relation $\sim$ on X by $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$, and let $X/\sim$ denote the quotient space. Show that f induces a well-defined map $g : X/\sim\ \to Y$ described by $g([x]) = f(x)$.

(iv) Show that g is continuous.

(v) Show that g is bijective and hence admits an inverse $g^{-1} : Y \to X/\sim$.

(vi) Show that the inverse of g is continuous, and hence g is a homeomorphism.

(vii) Show that f is open if and only if π is.

(viii) As sets, let $X = Y = [0, 1]$. Equip X with the standard topology. Let $f : X \to Y$ be defined by

$$f(x) = \begin{cases} 0 & 0 \le x \le 1/2 \\ 2x - 1 & 1/2 \le x \le 1. \end{cases}$$

Equip Y with the quotient topology induced by f. Show that f is continuous but not open. Conclude that the corresponding projection map $\pi : X \to X/\sim$ is not open.

2. Recall that an **abelian semigroup** is a set S together with a binary operation $* : S \times S \to S$ such that $s_1 * s_2 = s_2 * s_1$ for each $s_1, s_2 \in S$. We say that S has a **unit** if there is an element $e \in S$ such that $e * s = s$ for each $s \in S$. Theorem 4.6.2 says that $\mathbb{K}$ is an abelian semigroup with unit given by the unknot.

(i) If S has a unit, show that it is unique. (That is, if e_1, e_2 are two units for S, show that $e_1 = e_2$.)

(ii) We say that an element $s \in S$ **divides** another element $r \in S$, if there is an element $t \in S$ such that $s * t = r$. Show that the unknot divides every knot.

(iii) We say that an element s is **prime in** S if whenever s divides a product $a * b$, either s divides a or s divides b. Show that if K_P is a prime knot, then K_P is a prime element of $\mathbb{K}$.

(iv) We say that a non-unit $s \in S$ is **irreducible** if whenever $s = s_1 * s_2$ for some $s_1, s_2 \in S$, either $s_1 = e$ or $s_2 = e$. Show that every prime number is irreducible in $(\mathbb{N}_{>0}, \cdot)$.

(v) Show that every prime knot is irreducible in $\mathbb{K}$.

(vi) We say that an abelian semigroup with unit S has unique factorization if for each element $s \in S$ there are irreducible elements $s_1, \ldots, s_n \in S$ such that

$$s = e * s_1 * \cdots * s_n$$

and this representation is unique in the sense that if

$$s = e * t_1 \cdots * t_m$$

for some $t_1, \ldots, t_m \in S$, then $m = n$ and there is a bijection $\phi : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that $t_{\phi(i)} = s_i$ for each i. Why does $\mathbb{K}$ have unique factorization?

3. This exercise supplements the proof of Lemma 4.7.1 in Cromwell. Let $v = (v_1, v_2, v_3)$ be a vector in $\mathbb{R}^3$ such that $v_3 \neq 0$. If $H_+ = \{x_3 > 0\} \subset \mathbb{R}^3$ and $H_- = \{x_3 < 0\} \subset \mathbb{R}^3$, then either $v \in H_+$ or $v \in H_-$.

(i) If $v \in H_\pm$, show that there is a unique linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying $L(e_1) = e_1, L(e_2) = e_2$, and $L(v) = \pm e_3$.

(ii) Conclude that L is the identity on the x_1x_2-plane, and sends v to a vector perpendicular to this plane.

(iii) Show that L is orientation preserving.

(iv) Show that there is an isotopy from the identity map to L.

4. Cromwell 4.11.2

5. Cromwell 4.11.7

6. Cromwell 4.11.8

7. Cromwell 4.11.9