Knot Theory Seminar
Problem Set #8
Due Wednesday, July 8

1. Recall that an embedding of a topological space S into another space X is a continuous map $i : S \to X$ which induces a homeomorphism of S onto $i(S)$ when $i(S)$ is equipped with the subspace topology.

(i) If S is locally Euclidean of dimension n, show that $i(S)$ is as well.

(ii) If S is Hausdorff, show that $i(S)$ is as well.

(iii) If S is a manifold with boundary, show that $i(S)$ is as well.

(iv) If S is a manifold with boundary, is it true that $\delta(i(S)) = i(\delta S)$?

2. Show that the product of compact spaces is compact.

3. Define an equivalence relation on $X = [0,1] \times [0,1]$ by the rule $(x_1, y_1) \sim (x_2, y_2)$ if and only if one of the following conditions is satisfied

- $x_1 = x_2, y_1 = 0, y_2 = 1$
- $x_1 = x_2, y_1 = 1, y_2 = 0$
- $y_1 = y_2, x_1 = 0, x_2 = 1$
- $y_1 = y_2, x_1 = 1, x_2 = 0$.

Show that $X/\sim$ is homeomorphic to the two-dimensional torus T with big radius R and little radius r satisfying $0 < r < R$.

4. A group is a set G together with a binary operation $* : G \times G \to G$ such that

(a) there is an element $e \in G$ called an **identity element** such that $e * g = g * e = g$ for each $g \in G$

(b) for each element $g \in G$, there is an element $h \in G$ called an **inverse of** g such that $g * h = h * g = e$.

(c) the operation $*$ is associative in the sense that $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$ for each $g_1, g_2, g_3 \in G$.

Check whether the following are groups or not.

(i) $(\mathbb{Z}, +)$

(ii) $(\mathbb{Z}, \cdot)$

(iii) $(\mathbb{N}, +)$

(iv) $(\mathbb{Q}, +)$

(v) $(\mathbb{Q}, \cdot)$

(vi) $(\mathbb{Q} \setminus 0, \cdot)$

(vii) $(\mathbb{R}_{>0}, \cdot)$

(viii) $(\mathbb{R} \setminus 0, \cdot)$

(ix) The set of bijections of $\{1, \ldots, n\}$ onto itself together with function composition $\circ$.

(x) For a set X, the set of bijections $\mathrm{Perm}(X)$ of X onto itself together with function composition.

(xi) The set $M_n(\mathbb{R})$ of n-by-n matrices with coefficients in $\mathbb{R}$ together with matrix multiplication.

(xii) The set $GL(n, \mathbb{R})$ of invertible matrices of size n-by-n with coefficients in $\mathbb{R}$ together with matrix multiplication.

(xiii) The set $SL(n, \mathbb{R})$ of real matrices of size n-by-n with determinant 1 together with matrix multiplication.

5. Let G be a group.

(i) Show that the identity of G is unique.

(ii) Show that the inverse of an element g is unique.

6. A group $(G, *)$ is called **abelian** if the operation $*$ is commutative in the sense that $g_1 * g_2 = g_2 * g_1$ for each $g_1, g_2 \in G$. Which of the groups above are abelian?

7. A **homomorphism of groups** ϕ from $(G, *)$ into $(H, \cdot)$ is a map of sets $\phi : G \to H$ which satisfies $\phi(g_1 * g_2) = \phi(g_1) \cdot \phi(g_2)$.

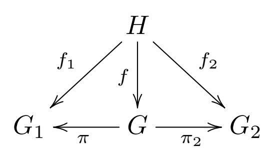
(i) Show that the determinant map $\det : GL(n, \mathbb{R}) \to \mathbb{R} \setminus 0$ is a group homomorphism.

(ii) Show that $\phi : \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(x) = 2x$ is a group homomorphism.

(iii) Show that $\phi : \mathbb{R} \to \mathbb{R}_{>0}$ defined by $\phi(x) = e^x$ is a group homomorphism.

(iv) Show that any linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ is a group homomorphism.

8. If $(G_1, *_1)$ and $(G_2, *_2)$ are groups, equip the Cartesian product $G_1 \times G_2$ with the structure of a group. As an added challenge, show that the group structure on $G = G_1 \times G_2$ satisfies the following universal property: There are homomorphisms $\pi_i : G \to G_i$ such that if $f_1 : H \to G_1$ and $f_2 : H \to G_2$ are group homomorphisms, then there is a unique homomorphism $f : H \to G$ such that the following diagram commutes



9. A group homomorphism is called an **isomorphism** if it admits an inverse which is also a homomorphism of groups. A group is called **free abelian group** if it is isomorphic to $\mathbb{Z}^k = \mathbb{Z} \times \cdots \times \mathbb{Z}$ for some k. The number k is called the **rank** of G. Free abelian groups are like vector spaces, except the scalars come from the integers $\mathbb{Z}$ instead of a field.

(i) Let $v_1, \ldots, v_k$ be k linearly independent vectors in $\mathbb{R}^n$. Show that the set of $\mathbb{Z}$-linear combinations of $v_1, \ldots, v_k$ forms a free abelian group of rank k.

(ii) Let X be a set of cardinality k. Show that the set of functions $X \to \mathbb{Z}$ enjoys the structure of a free abelian group of rank k.

(iii) Let X be a set of cardinality k. Show that the set of finite formal $\mathbb{Z}$-linear combinations of elements of X enjoys the structure of a free abelian group of rank k.

(iv) Let G be a free abelian group of rank k. Endow the set G^* of homomorphisms $G \to \mathbb{Z}$ with the structure of a free abelian group, called the group dual to G.

10. The kernel of a group homomorphism $\phi : G \to H$ is $\ker \phi = \{g \in G : \phi(g) = e_H\}$ where e_H is the identity element from H.

(i) Show that the kernel $\ker \phi$ enjoys the structure of a group when the operation from G is restricted to $\ker \phi$. We say that $\ker \phi$ is a **subgroup of** G. (Note that you must in particular check that the product of two elements of the kernel is another element of the kernel.)

(ii) Exhibit $SL(n, \mathbb{R})$ as the kernel of some homomorphism ϕ.

11. Let G be a group. A subset H of G is called a **subgroup of** G if H is a group itself when the operation from G is restricted to H. Show that the following are equivalent for a nonempty subset H of G

- H is a subgroup of G.
- If x, y belong to H, then xy belongs to H and x^{-1} belongs to H.
- If x, y belong to H, then xy^{-1} belongs to H.

12. Let G be an *abelian* group and H a subgroup of G. Then we may form what is called the **quotient group** G/H in the following manner.

(i) For $g \in G$, let gH denote the subset of G given by $gH = \{gh : h \in H\}$. Such a subset is called a **left coset of** G. For $g \in G$, show that the map $\phi_g : H \to gH$ defined by $\phi_g(h) = gh$ is a bijection, and conclude that all left cosets have the same cardinality. Our goal is now to endow the collection of left-cosets with the structure of a group.

(ii) Show that $g_1 H = g_2 H$ if and only if $g_2^{-1} g_1 \in H$.

(iii) Define a relation on G by $g_1 \sim g_2$ if and only if $g_1 H = g_2 H$. Show that $\sim$ is an equivalence relation.

(iv) Show that $[g] = gH$.

(v) Let G/H denote the set of equivalence classes under this equivalence relation. Show that the binary operation defined on G/H by $[g_1] * [g_2] = [g_1 g_2]$ is well-defined.

(vi) Show that G/H acquires the structure of an abelian group when endowed with this operation.

We remark that in general, we may only form the quotient group by a so-called *normal subgroup*, but when G is abelian, it turns out that all subgroups are normal, so we can always form the quotient group in this case.