Knot Theory Seminar
Problem Set #9
Solutions to selected problems

1. Let X be a simplicial complex. Let $C_\ell(X, \mathbb{Z}/2\mathbb{Z})$ denote the free abelian group of formal linear combinations of ℓ-simplices of X with coefficients in $\mathbb{Z}/2\mathbb{Z}$. We call $C_\ell(X, \mathbb{Z}/2\mathbb{Z})$ the group of **ℓ-chains**. Define the boundary operator $\partial_\ell : C_\ell(X, \mathbb{Z}/2\mathbb{Z}) \to C_{\ell-1}(X, \mathbb{Z}/2\mathbb{Z})$ as in class. In particular, if $[v_i, v_j]$ denotes the oriented 1-simplex with vertices v_i and v_j, then

$$\partial_1([v_i, v_j]) = v_j - v_i \equiv v_j + v_i \mod 2.$$

Let Z_1 denote the kernel of ∂_1, and call the elements of Z_1 **cycles**. On the other hand, a **circuit in** X is a sum of the form $\sum_{i=0}^{n-1}[v_i, v_{i+1}]$ such that

(a) $v_0 = v_n$,

(b) No other two v_i and v_j are equal.

Let S denote the subset of $C_1(X, \mathbb{Z}/2\mathbb{Z})$ consisting of all linear combinations of circuits. Prove that $S = Z_1$. (Hint: First prove that every circuit is a cycle. To prove the reverse inclusion, use strong induction on the number of nonzero terms in an 1-chain that is a cycle.)

Proof. We first prove the inclusion $S \subset Z_1$. If $\sum_{i=0}^{n-1}[v_i, v_{i+1}]$ is a circuit, then

$$\partial\left(\sum_{i=0}^{n-1}[v_i, v_{i+1}]\right) = \sum_{i=0}^{n-1}(v_{i+1} - v_i) = v_n - v_0 = 0$$

where the last equality comes from the fact that $v_n = v_0$ for a circuit. Since ∂ is linear, it follows that any linear combination of circuits is also a cycle. Hence $S \subset Z_1$. It remains to show the reverse inclusion.

We prove the inclusion $Z_1 \subset S$ by strong induction on the number of nonzero terms in a cycle $C = \sum_{i=1}^{n} e_i$. For an edge e_i, let v_i^1 and v_i^2 denote the first and last vertices of the edge respectively. There are no cycles with one or two nonzero terms, since the vertices of an edge must be distinct.

Suppose that $n = 3$. Then we find that

$$0 = \partial(C) = v_1^2 - v_1^1 + v_2^2 - v_2^1 + v_3^2 - v_3^1 \equiv v_1^2 + v_1^1 + v_2^2 + v_2^1 + v_3^2 + v_3^1.$$

One vertex besides v_1^1 itself must be equal to v_1^1 in order for the sum to be zero. This vertex cannot be v_1^2, since e_1 is an edge with distinct vertices. So without loss of generality, we may assume that $v_3^2 = v_1^1$. In this case, the sum reduces to

$$v_1^2 + v_2^2 + v_2^1 + v_3^1.$$

One of the vertices besides v_1^2 itself must be equal to v_1^2 in order for the sum to be zero. It cannot be v_3^1, or else the edge e_3 would not be distinct from the edge e_1 (which would cause C to have only one nonzero term). So without loss of generality, we may assume that $v_2^2 = v_1^2$. In this case, or sum reduces again and we find that $v_2^1 = v_3^1$. It follows that C satisfies conditions (a) and (b) of the definition of a circuit. This completes the base case when $n = 3$.

Suppose now that $n > 3$ and that the inductive hypothesis holds for all cycles with less than n nonzero terms. Let $C = \sum_{i=1}^{n} e_i$ be a cycle. If there is a proper nonzero subsum $C' = \sum_{j=1}^{k} e_{i_j}$ which is a cycle, then by the inductive hypothesis, C' is a linear combination of circuits and so is $C - C'$, and hence so is C. So we may assume that there are no nonzero proper subsums that are cycles. Because C is a cycle, we have the equality

$$0 = \sum_{i=1}^{n}(v_i^2 + v_i^1).$$

Similar to the above reasoning, we may assume that $v_2^1 = v_1^2$. In this case, we have written

$$C = [v_1^1, v_1^2] + [v_1^2, v_2^2] + C'$$

where C' is a subsum of C. Because edges need to be distinct in C and have distinct vertices, we find that $v_1^1 \neq v_1^2$, $v_1^2 \neq v_2^2, v_1^1 \neq v_2^2$. Because C has no nonzero proper subsums that are cycles, we find that either $C' = 0$ or we must have $v_1^1 \neq v_2^2$. If $C' = 0$, then we find that the first vertex of the first edge of C is equal to the second vertex of the last edge of C, and hence C is a circuit. If C' is nonzero, then because C is a cycle, we have

$$0 = v_1^1 + v_2^2 + \partial(C')$$

Hence, without loss of generality, we may assume that $v_3^1 = v_2^2$ and hence we have written C as

$$C = [v_1^1, v_1^2] + [v_1^2, v_2^2] + [v_2^2, v_3^2] + C''.$$

Again either $C'' = 0$ or $v_3^2 \neq v_1^1$. If C'' is zero, then C is a circuit. Otherwise, we may continue in this way to write C as

$$C = [v_1^1, v_1^2] + [v_1^2, v_2^2] + \cdots + [v_{n-1}^2, v_n^2].$$

Because C is a cycle, we find that

$$0 = v_1^1 + v_n^2.$$

Hence $v_1^1 = v_n^2$, and we find that C is a circuit. This completes the inductive step and the proof. $\square$