Knot Theory Seminar
Problem Set #9
Due Tuesday, July 14

1. Let X be a simplicial complex. Let $C_\ell(X, \mathbb{Z}/2\mathbb{Z})$ denote the free abelian group of formal linear combinations of ℓ-simplices of X with coefficients in $\mathbb{Z}/2\mathbb{Z}$. We call $C_\ell(X, \mathbb{Z}/2\mathbb{Z})$ the group of **ℓ-chains**. Define the boundary operator $\partial_\ell : C_\ell(X, \mathbb{Z}/2\mathbb{Z}) \to C_{\ell-1}(X, \mathbb{Z}/2\mathbb{Z})$ as in class. In particular, if $[v_i, v_j]$ denotes the oriented 1-simplex with vertices v_i and v_j, then

$$\partial_1([v_i, v_j]) = v_j - v_i \equiv v_j + v_i \mod 2.$$

Let Z_1 denote the kernel of ∂_1, and call the elements of Z_1 **cycles**. On the other hand, a **circuit in** X is a sum of the form $\sum_{i=0}^{n-1}[v_i, v_{i+1}]$ such that

(a) $v_0 = v_n$,

(b) No other two v_i and v_j are equal.

Let S denote the subset of $C_1(X, \mathbb{Z}/2\mathbb{Z})$ consisting of all linear combinations of circuits. Prove that $S = Z_1$. (Hint: First prove that every circuit is a cycle. To prove the reverse inclusion, use strong induction on the number of nonzero terms in an 1-chain that is a cycle.)

2. Denote by $\Delta^n = [e_0, \ldots, e_n]$ the standard n-simplex, with vertices at $e_0, \ldots, e_n$. Recall that the boundary operator is linear and satisfies

$$\partial([e_0, \ldots, e_n]) = \sum_{i=0}^{n} (-1)^i [e_0, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n].$$

(i) Show that $\partial\partial([e_0, \ldots, e_n]) = 0$.

(ii) In general, an n-simplex in a simplicial complex takes the form $[v_0, \ldots, v_n]$ where $v_0, \ldots, v_n$ are the vertices of the simplex, listed in the order compatible with the orientation. Argue that your argument for (i) shows that $\partial\partial([v_0, \ldots, v_n]) = 0$, and hence that $\partial^2 \equiv 0$.

(iii) Conclude that $B_\ell(X, \mathbb{Z}) \subset Z_\ell(X, \mathbb{Z})$, and hence the quotient group $H_\ell(X, \mathbb{Z})$ is well-defined.

3. The goal of this exercise is to show that $H_1(S^2, \mathbb{Z}/2\mathbb{Z}) = 0$.

(i) Let $X = \partial\Delta^3$ denote the boundary of the standard 3-simplex. Note that X is homeomorphic to the unit sphere S^2. Compute the dimensions of the vector spaces $C_\ell(X, \mathbb{Z}/2\mathbb{Z})$ for $\ell = 0, 1, 2$.

(ii) Show that $\dim Z_1(X, \mathbb{Z}/2\mathbb{Z}) = 3$. (Hint: Use Theorem 6.1.2)

(iii) Show that $\dim B_1(X, \mathbb{Z}/2\mathbb{Z}) = 3$ (Hint: Show that $Z_1 = B_1$)

(iv) Show that $H_1(X, \mathbb{Z}/2\mathbb{Z}) = 0$.

(v) Conclude that $H_1(S^2, \mathbb{Z}/2\mathbb{Z}) = 0$.

4. Compute $H_1(T, \mathbb{Z}/2\mathbb{Z})$ where T is the torus. (Hint: The answer is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Use a suitable simplicial complex as is done in Problem 3. Try to modify the the standard representation of the torus as a square with edges identified to give a simplicial complex homeomorphic to T. Remember that the edges and vertices of a 2-simplex must be distinct.)

5. Compute a Seifert matrix for the trefoil, and use this to compute the determinant and the signature of the trefoil.

6. Compute a Seifert matrix for the figure-8 knot, and use this to compute the determinant and the signature of the trefoil.

7. Compute a Seifert matrix for the Hopf link, and use this to compute the determinant and the signature of the Hopf link.

8. A **chain complex of vector spaces** $C_\bullet$ is a sequence of vector spaces $\{C_n\}$ together with linear maps $\partial_n : C_n \to C_{n-1}$ such that $\partial_{n-1}\partial_n = 0$ for each n. For a chain complex, the n-th homology group $H_n(C_\bullet)$ is well-defined and given by

$$H_n(C_\bullet) := \frac{\ker \partial_n}{\mathrm{im}\partial_{n+1}}$$

Show that if

$$0 \to C_n \to C_{n-1} \to \cdots \to C_2 \to C_1 \to C_0 \to 0$$

is a chain complex of vector spaces, then

$$\sum_{i=0}^{n} (-1)^i \dim C_i = \sum_{i=0}^{n} (-1)^i \dim H_i(C_\bullet).$$

The above quantity is called the **Euler characteristic** of the chain complex, and is denoted

$$\chi(C_\bullet) = \sum_{i=0}^{n} (-1)^i \dim C_i.$$

9. Let F be an orientable surface. Show that the Euler characteristic $\chi(F)$ is equal to the Euler characteristic of $C_\bullet(X, \mathbb{Z}/2\mathbb{Z})$ for a suitable triangulation X of F.

10. Our goal is to show that for a knot K, there are $2g(F)$ rows and columns in a Seifert matrix, where F is a Seifert surface. Remember that a Seifert surface for K is an orientable surface with boundary whose boundary is equal to K.

(i) If X is a triangulation of a surface with at least one boundary component, argue that $H_2(X, \mathbb{Z}/2\mathbb{Z}) = 0$.

(ii) Say that a simplicial complex X is **path connected** if for each pair of vertices, there is a sequence of edges connecting the two vertices. Show that if X is path-connected, then $\dim H_0(X, \mathbb{Z}/2\mathbb{Z}) = 1$.

(iii) Conclude that if F is path-connected, then $\dim H_0(F, \mathbb{Z}/2\mathbb{Z}) = 1$.

(iv) Show that $\dim H_1(F, \mathbb{Z}/2\mathbb{Z}) = 2g(F)$. (Hint: Use Problems 8 and 9.)