On vertex operator algebras, their representations, and corresponding two-dimensional conformal geometry

A Thesis

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Chapter 1

Introduction

1.1 Historical remarks

The immediate precursor of a vertex operator algebra is the notion of a vertex algebra, first introduced in 1986. In particular, Borcherds [Bo] constructed an algebraic structure on a certain Fock space (a space of particle states) associated with a root lattice of a Kac-Moody algebra. He called these constructions vertex algebras, because they were related to particle interactions at a "vertex," employing concepts in quantum mechanics and quantum field theories, as well as the representation theory of infinite-dimensional Lie algebras.

Roughly speaking a vertex algebra is a quadruple $(V, Y, \mathbb{1}, d)$ where V is a $\mathbb{Z}$-graded vector space,

$$Y(\cdot, x) : V \longrightarrow (\operatorname{End} V)[[x, x^{-1}]]$$

is a linear map from V to the space of formal Laurent series in x with coefficients in $\operatorname{End} V$, $\mathbb{1}$ is a special vector, which serves as a type of identity for $Y(\cdot, x)$, and d is an endomorphism of V. These data are supposed to satisfy certain axioms, among which imply that the multiplication $Y(\cdot, x)$ is "almost" commutative and "almost" associative, in some sense. The axioms have been simplified and modernized over the years, especially due to the work of Frenkel, Lepowsky, and Meurman [FLM88].

Surprisingly, vertex algebras quickly found a place in group theory, answering a conjecture concerning sporadic groups. The largest of the sporadic groups (the Monster) exhibited interesting connections with a certain modular function, denoted j, which led Conway and Norton [CN] to conjecture that an infinite-dimensional $\mathbb{Z}$-graded representation of this group could be constructed in such a way that the graded dimension coincided with the Laurent series expansion of j. Borcherds [Bo] conjectured that such a representation could acquire the structure of a vertex algebra, and in 1984, Frenkel, Lepowsky, and Meurman [FLM84] showed that this was indeed the case, constructing the notable *Moonshine module* on which the Monster group acts by symmetries. This construction demonstrated the rich, diverse connections among number theory, group theory, Lie algebras, and physics.

With the construction of the Moonshine module, Frenkel, Lepowsky, and Meurman [FLM88] modified Borcherds's definition of a vertex algebra to give the notion of a vertex operator algebra. The modification included the addition of an element ω of weight two whose vertex operator $Y(\omega, z)$ gives a representation of the Virasoro algebra, that is, the complex Lie algebra spanned by elements L_n for $n \in \mathbb{Z}$ and central element c subject to the relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

This algebra is a central extension of the Witt algebra, which consists of the meromorphic vector fields on the unit circle S^1 in $\mathbb{C}$ and which plays a key role in the study of conformal field theory and string theory.

The algebraic formulation of vertex algebras (and consequently vertex operator algebras) was revised and organized over the years, perhaps culminating in Li's results on *local systems* of vertex operators. In [L96], Li studied equivalent formulations of the algebraic axioms of a vertex algebra, generalizing Dong and Lepwosky's result [DL] which states that the Jacobi identity axiom may be replaced by commutativity for vertex algebras under mild assumptions. From this axiomatic approach, Li then went on to show that any local system of vertex operators on a vector space M carries a natural vertex algebra structure. These results allow for easier construction of vertex algebras, as it is merely sufficient to find a local system of vertex operators.

A major development in the representation theory of vertex operator algebras was the construction of a functor Λ, found originally in [Z], from the category of vertex operator algebras to the category of associative algebras with unit. This functor is useful because it gives rise to a bijective correspondence between the irreducible weak-admissible modules for a vertex operator algebra V and irreducible modules for its corresponding associative algebra $\Lambda(V)$. The corresponding algebra $\Lambda(V)$ is often called *Zhu's algebra* in the literature, and it has played a key role in the classification of rational and regular vertex operator algebras [DLM97a, DLM97b, FZ, W].

Finally a work of Huang [H97] established an equivalence between vertex operator algebras and a certain construction resulting from the study of two-dimensional conformal geometry. It was known that vertex operator algebras played a role in mathematical physics and string theory, but it was only until Huang's work that the role of the sewing equation and two-dimensional conformal geometry in the formulation of vertex operator algebras was made rigorous. In simplest terms, Huang showed that vertex operator algebras arise as algebraic structures governed by the interactions of closed vibrating strings in space time. More specifically, he demonstrated that vertex operator algebras are categorically equivalent to (partial) algebras that arise over a certain partial operad consisting of moduli spaces of Riemann spheres with punctures and local coordinates together with a composition given by the sewing equation. Huang's work extended the already fascinating connections with vertex operator algebras even further, outlining rigorously the role of two-dimensional conformal geometry and the sewing equation in their formulations.

1.2 Overview

In Chapter 2, we follow [LL] to introduce the notion of a vertex operator algebra from a purely formal algebraic standpoint. We first study formal calculus, developing the necessary notation and theory, specifically the *delta function identities*, which are fundamental tools to study the algebraic relations involved in the theory of vertex operator algebras. After discussing the more general notion of a vertex algebra, we introduce a vertex operator algebra as a type of vertex algebra with a distinguished *Virasoro vector* ω. We then provide a non-trivial, yet basic example which we call the *space of one free boson.*

The next chapter shows how vertex operator algebras are a symmetric monoidal category when equipped with a tensor product structure. For the reader less familiar with the language of category theory, we have included several sections on this material, introducing the notions of categories, functors, monoidal categories, and monoidal functors. We then provide basic examples of monoidal categories through studying ring modules, and more specifically, associative algebras. We complete Chapter 3 by developing a tensor product structure (or monoidal structure) on the category of vertex operator algebras, and we show how this monoidal structure is symmetric.

Chapter 4 is a study of the functorial properties of Zhu's algebra (first constructed in [Z]) especially with respect to the monoidal structures of the categories at play. Our main result is showing how this functor respects the symmetric monoidal structures of these categories. We then summarize other properties of this and related functors important to the representation theory of vertex operator algebras.

The last chapter is an introduction to the geometry underlying the theory of vertex operator algebras, following closely Huang's work [H97]. We leave out some of the more intricate details of Huang's original work, and we are content simply to motivate and introduce the geometric structures necessary to develop the notion of a *geometric vertex operator algebra*. It is lastly stated that this geometric construction is equivalent to the algebraic construction developed in the first chapter, that is, the categories of vertex operator algebras and geometric vertex operator algebras are isomorphic.

1.3 Notations

$\mathbb{C}$	Field of complex numbers
$\mathbb{C}^\times$	Group of nonzero elements of $\mathbb{C}$
$\mathbb{N}$	Set of nonnegative integers
$\mathbb{Q}$	Field of rational numbers
$\mathbb{R}$	Field of real numbers
S_n	Symmetric group on n letters
$\mathbb{Z}$	Ring of integers
$\mathbb{Z}_+$	Set of positive integers

Chapter 2

Vertex operator algebras

Following [LL], we introduce the notion of a vertex operator algebra from a purely algebraic standpoint. While this algebraic approach is perhaps easier to grasp quickly, it fails to capture how the geometry of propagating strings motivates the definition. Therefore, after the algebraic approach of this chapter and the next, we supply the geometric motivation in the last.

An important tool in the study of vertex operator algebras is *formal calculus*, which consists of the study of formal power series with coefficients in some vector space as an algebraic-like object with a natural partial multiplication structure. The formal delta function $\delta(x)$ and formal exponentiation function play key roles, as does the binomial expansion convention

$$(x_1 + x_2)^r = \sum_{k \in \mathbb{N}} \binom{r}{k} x_1^{r-k} x_2^k$$

which is an extension of the convention for complex numbers. The study of formal calculus helped to condense some of the relations of the algebraic structure for vertex operator algebras to the single Jacobi identity

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)Y(u, x_1) \\
&= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2),
\end{aligned}$$

which uses in a fundamental way the properties of the formal delta function. The next two sections are designed to highlight and offer a proof of these properties, as well as to discuss other important results within formal calculus.

2.1 Formal Calculus

We work exclusively over the complex numbers $\mathbb{C}$. In particular, all vector spaces are assumed to be complex. We typically use the variables x, y, t to denote commuting independent formal variables.

For a vector space V, we let $V[[x, x^{-1}]]$ denote the space of *formal Laurent series* in x with coefficients in V, that is,

$$V[[x, x^{-1}]] = \left\{ \sum_{n \in \mathbb{Z}} v_n x^n : v_n \in V \right\}.$$

This space is a vector space endowed with the usual operations of addition and scalar multiplication (from $\mathbb{C}$). If $Y : V \to W$ is a linear map of vector spaces, then Y induces a linear map

$$Y : V[[x, x^{-1}]] \longrightarrow W[[x, x^{-1}]]$$

which we still denote by Y. Two important subspaces of $V[[x, x^{-1}]]$ are the space

$$V[[x]] = \left\{ \sum_{n \in \mathbb{N}} v_n x^n : v_n \in V \right\}$$

of *formal power series* in x with coefficients in V and the space

$$V((x)) = \left\{ \sum_{n \in \mathbb{Z}} v_n x^n : v_n \in V, v_n = 0 \text{ for } n \text{ sufficiently negative} \right\}$$

of *truncated formal Laurent polynomials* in x with coefficients in V.

Remark 2.1.1. Even though the space of polynomials in x with coefficients in V has a multiplication structure, the above space of formal Laurent series does not in general inherit this well-defined multiplication structure. In particular, the reader should consider how to define the product

$$\left(\sum_{n \in \mathbb{Z}} x^n \right) \left(\sum_{n \in \mathbb{Z}} x^n \right)$$

in $\mathbb{C}[[x, x^{-1}]]$, using the methods of polynomial multiplication. (The problem is that too many coefficients will be "lumped" into a single power of x, thus giving an ill-defined coefficient for this power of x.)

Nevertheless, we wish to define a product structure on this space. In particular, we must define a partial multiplicative structure on the space $(\operatorname{End} V)[[x, x^{-1}]]$ where V is a vector space.

The idea will be to define multiplication in the usual way, but only to restrict our attention to those products which are well-defined. In particular, we do not want products with the type of infinite lumping found in the ill-defined product of Remark 2.1.1.

A family of operators $\{f_i\}_{i\in I}$ in $\operatorname{End} V$ is said to be *summable* if for each $v \in V$, the element $f_i v$ is zero for all but a finite number of i in the indexing set I. If this is the case, we write the "sum" operator as

$$\sum_{i\in I} f_i : V \longrightarrow V$$
$$v \mapsto \sum_{i\in I} f_i(v).$$

Let $\{F_i(x)\}_{i\in I}$ be a family in $(\operatorname{End} V)[[x, x^{-1}]]$ and for each $i \in I$, set

$$F_i(x) = \sum_{n\in\mathbb{Z}} f_i(n)x^n.$$

We say that the sum $\sum_{i\in I} F_i(x)$ *exists* if for each $n \in \mathbb{Z}$, the family $\{f_i(n)\}_{i\in I}$ is summable. If this is the case, we then set

$$\sum_{i\in I} F_i(x) = \sum_{n\in\mathbb{Z}} \left(\sum_{i\in I} f_i(n) \right) x^n.$$

We say that the product $F_1(x)\cdots F_r(x)$ *exists* if for each $n \in \mathbb{Z}$, the family

$$\{f_1(n_1)\cdots f_r(n_r)\}_{n_1+\cdots+n_r=n}$$

is summable. If this is the case, we then set

$$F_1(x)\cdots F_r(x) = \sum_{n\in\mathbb{Z}} \left(\sum_{n_1+\cdots+n_r=n} f_1(n_1)\cdots f_r(n_r) \right) x^n.$$

Remark 2.1.2. These elementary principles and definitions play an important role in the development of formal calculus. The notion of nonexistent product should not be taken lightly, and the careful student of formal calculus will do well to pay close attention to how these definitions play a role when attempting exercises or computations.

Key to the axioms of vertex operator algebras is the delta function and the identities in which it appears. The *formal delta function* $\delta(x)$ is the formal series defined by

$$\delta(x) = \sum_{n\in\mathbb{Z}} x^n \in \mathbb{C}[[x, x^{-1}]].$$

Although suppressed, the coefficients of each term in the series are taken to be the identity element 1 of the field $\mathbb{C}$.

If V is a complex vector space, then we have a natural injection

$$\mathbb{C}[[x,x^{-1}]] \longrightarrow (\operatorname{End} V)[[x,x^{-1}]]$$
$$\sum_{n\in\mathbb{Z}} c_n x^n \mapsto \sum_{n\in\mathbb{Z}} (c_n \cdot \mathrm{id}_V) x^n,$$

and we regard $\delta(x)$ as an element of $(\operatorname{End} V)[[x,x^{-1}]]$ in this way.

Remark 2.1.3. Although the label *function* suggests that $\delta(x)$ can be evaluated for some complex values of x, this is certainly not the case. In fact, $\delta(x)$ cannot be evaluated for any value of x, and this "function" should instead be considered a formal series. The x simply denotes the letter we are using to denote the formal variable.

Remark 2.1.4. When working with the delta function, it is important to keep in mind the rules for existent and non-existent products, as discussed above. For example, it is clear that the product

$$\delta(x)\delta(x)$$

mentioned in Remark 2.1.1 does not exist. This example shows how challenging it could be to demonstrate that a product of two formal series is defined: we must check whether the coefficient of each power of x acts like a finite sum. This example also shows that having a well-defined product is a very restrictive condition: only one coefficient can ruin the whole product.

We use the following *binomial expansion convention*. For a complex number $r \in \mathbb{C}$, we let $(x_1+x_2)^r$ be the formal series

$$(x_1+x_2)^r = \sum_{k\in\mathbb{N}} \binom{r}{k} x_1^{r-k} x_2^k$$

in $\mathbb{C}[[x_1,x_1^{-1},x_2]]$ where

$$\binom{r}{k} = \frac{r(r-1)\cdots(n-k+1)}{k!}.$$

We also have a formal derivative, defined in the obvious way. If V is a complex vector space and $v(x) = \sum_{n\in\mathbb{Z}} v_n x^n$ is an element of $V[[x,x^{-1}]]$, then we define the *formal derivative* $\frac{d}{dx}v(x) = v'(x)$ to be the formal series

$$\frac{d}{dx}v(x) = \sum_{n\in\mathbb{Z}} n v_n x^{n-1}.$$

The formal partial derivatives $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$, ... are defined in similar fashions. It is easy to see that $\frac{d}{dx}$ acts as a linear operator on the space $V[[x, x^{-1}]]$, and more specifically, as a derivation. The same is true of the partial derivatives.

Another key series is the exponential function, which is an extension of the exponential function from the complex numbers.

Let S be an element of $x(\text{End}\, V)[[x]]$, that is, S has no constant term. As convention, we let S^0 denote the element id $\in (\text{End}\, V)[[x]]$ and set $S^n = S^{n-1} \cdot S$ for n positive. Then each S^n is a well-defined element of $(\text{End}\, V)[[x]]$ since S is truncated from below. Also the sum

$$e^S = \sum_{n \in \mathbb{N}} \frac{1}{n!} S^n$$

exists and is a well defined element of $(\text{End}\, V)[[x]]$, since S involves only positive powers of x.

Finally, we need *formal limits*. Let $\sum_{m,n \in \mathbb{Z}} F(m,n) x_1^m x_2^n$ be an element of $V[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$. We say that the limit

$$\lim_{x_1 \to x_2} \sum_{m,n \in \mathbb{Z}} F(m,n) x_1^m x_2^n$$

exists if for each $n \in \mathbb{Z}$ the family $\{F(m, n-m)\}_{m \in \mathbb{Z}}$ is summable, in which case, we set

$$\lim_{x_1 \to x_2} \sum_{m,n \in \mathbb{Z}} F(m,n) x_1^m x_2^n = \sum_{n \in \mathbb{Z}} \left(\sum_{m \in \mathbb{Z}} F(m, n-m) \right) x_2^n.$$

From the above definitions, we obtain the *Formal Taylor Theorem*.

Theorem 2.1.5. *Let $v(x)$ be an element of $V[[x, x^{-1}]]$. Then the equality*

$$e^{y \frac{d}{dx}} v(x) = v(x+y)$$

holds, and in particular, both expressions exist.

Proof. Write $v(x) = \sum_{n \in \mathbb{Z}} v_n x^n$. Then observe that

$$\begin{aligned} e^{y \frac{d}{dx}} v(x) &= \sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{Z}} \frac{y^k}{k!} \left(\frac{d}{dx} \right)^k v_n x^n \\ &= \sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{Z}} \frac{y^k}{k!} n(n-1) \cdots (n-k+1) v_n x^{n-k} \\ &= \sum_{n \in \mathbb{Z}} v_n \sum_{k \in \mathbb{N}} \binom{n}{k} y^k x^{n-k} \\ &= \sum_{n \in \mathbb{Z}} v_n (x+y)^n \end{aligned}$$

$$= v(x+y).$$

□

Exercise 2.1.6. Show that the product

$$\left(\sum_{n\geq 0} x^n\right)\left(\sum_{n\leq 0} x^n\right)$$

does *not* exist.

Exercise 2.1.7. Show that the product

$$\left(\sum_{n\geq 0} \frac{x^n}{2^n}\right)\left(\sum_{n\leq 0} x^n\right)$$

does *not* exist.

Exercise 2.1.8. If $f(x) \in V[x, x^{-1}]$, show that the product $f(x)\delta(x)$ *does* exist. Also show that

$$f(x)\delta(x) = f(1)\delta(x).$$

(Hint: First show that $x^n\delta(x) = \delta(x)$ for any $n \in \mathbb{Z}$.)

Exercise 2.1.9. Show that

$$x^{-1}\delta\left(\frac{y}{x}\right) = y^{-1}\delta\left(\frac{y}{x}\right) = y^{-1}\delta\left(\frac{x}{y}\right).$$

2.2 Delta function identities

The goal of this section is to address some important identities involving the formal delta function. These identities play a fundamental role in the formulation of the Jacobi identity axiom for vertex algebras and vertex operator algebras.

As usual, fix a complex vector space V. We use the variables x_0, x_1, x_2 to denote formal independent commuting variables.

Lemma 2.2.1. *For $m \in \mathbb{C}$ and $k \in \mathbb{N}$, we have*

$$(-1)^k\binom{k-m-1}{k} = \binom{m}{k}.$$

Proof. Note that

$$(-1)^k \binom{k-m-1}{k} = (-1)^k \frac{\prod_{j=0}^{k-1}(k-m-1-j)}{k!} = \frac{\prod_{j=0}^{k-1}(m-k+1+j)}{k!}.$$

Replacing $i = -j + k - 1$, we see that

$$\frac{\prod_{j=0}^{k-1}(m-k+1+j)}{k!} = \frac{\prod_{i=k-1}^{0}(m-i)}{k!} = \frac{\prod_{i=0}^{k-1}(m-i)}{k!} = \binom{m}{k}.$$

∎

Proposition 2.2.2. *In* $V[[x_0, x_0^{-1}, x_1, x_1^{-1}, x_2, x_2^{-1}]]$, *we have*

$$x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right). \tag{2.2.1}$$

Proof. Observe that

$$x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) = x_2^{-1}\sum_{n\in\mathbb{Z}} x_2^{-n}\left(x_1 - x_0\right)^n = \sum_{n\in\mathbb{Z}} x_2^{-n-1}\sum_{k\in\mathbb{N}}\binom{n}{k} x_1^{n-k}(-1)^k x_0^k = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{N}}(-1)^k\binom{n}{k} x_0^k x_1^{n-k} x_2^{-n-1}.$$

Now let $n = k - m - 1$, and use Lemma 2.2.1 to observe that

$$\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{N}}(-1)^k\binom{n}{k} x_0^k x_1^{n-k} x_2^{-n-1} = \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{N}}(-1)^k\binom{k-m-1}{k} x_0^k x_1^{-m-1} x_2^{m-k} = \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{N}}\binom{m}{k} x_0^k x_1^{-m-1} x_2^{m-k} = \sum_{m\in\mathbb{Z}} x_1^{-m-1}\sum_{k\in\mathbb{N}}\binom{m}{k} x_2^{m-k} x_0^k = x_1^{-1}\sum_{m\in\mathbb{Z}} x_1^{-m}\left(x_2 + x_0\right)^m$$

$$= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right),$$

as desired. ☐

We call (2.2.1) the *two-term delta function identity.* We also have the following *three-term delta function identity* (2.2.2), which requires some intermediary results.

Lemma 2.2.3. *We have*

$$(-1)^k\binom{-1}{k} = 1$$

for each $k \in \mathbb{N}$.

Proof. This follows from Lemma 2.2.1 with $m = k$. ☐

Lemma 2.2.4. *In $V[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$, we have*

$$(x_1-x_2)^{-m-1} - (-x_2+x_1)^{-m-1} = \frac{(-1)^m}{m!}\left(\frac{\partial}{\partial x_1}\right)^m x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)$$

for each $m \in \mathbb{N}$.

Proof. We prove the claim by induction on the letter m. For $m = 0$, we apply Lemma 2.2.3 to obtain

$$\begin{aligned}(x_1-x_2)^{-1} - (-x_2+x_1)^{-1} &= \sum_{k\in\mathbb{N}}\binom{-1}{k}(-1)^k x_1^{-1-k}x_2^k - \sum_{k\in\mathbb{N}}\binom{-1}{k}(-1)^{-1-k}x_1^k x_2^{-1-k}\\ &= \sum_{k\in\mathbb{N}} x_1^{-1-k}x_2^k + \sum_{k\in\mathbb{N}} x_1^k x_2^{-1-k}\\ &= x_2^{-1}\delta\left(\frac{x_1}{x_2}\right),\end{aligned}$$

as desired.

Now assume that the claim holds for some nonnegative integer m. Then observe that

$$\begin{aligned}\frac{(-1)^{(m+1)}}{(m+1)!}\left(\frac{\partial}{\partial x_1}\right)^{(m+1)} x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) &= \frac{-1}{m+1}\frac{\partial}{\partial x_1}\left(\frac{(-1)^m}{m!}\left(\frac{\partial}{\partial x_1}\right)^m x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\right)\\ &= \frac{-1}{m+1}\frac{\partial}{\partial x_1}\left((x_1-x_2)^{-m-1} - (-x_2+x_1)^{-m-1}\right)\\ &= \frac{-1}{m+1}\left((-m-1)(x_1-x_2)^{-m-2} - (-m-1)(-x_2+x_1)^{-m-2}\right)\\ &= (x_1-x_2)^{-(m+1)-1} - (-x_2+x_1)^{-(m+1)-1},\end{aligned}$$

thus completing the inductive step and the proof. □

Proposition 2.2.5. *In $V[[x_0, x_0^{-1}, x_1, x_1^{-1}, x_2, x_2^{-1}]]$, we have*

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right). \tag{2.2.2}$$

Proof. Observe that

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) = \sum_{n\in\mathbb{Z}} x_0^{-n-1}\left((x_1 - x_2)^n - (-x_2 + x_1)^n\right).$$

Letting $m = -n - 1$, we see that

$$\sum_{n\in\mathbb{Z}} x_0^{-n-1}\left((x_1 - x_2)^n - (-x_2 + x_1)^n\right) = \sum_{m\in\mathbb{Z}} x_0^m((x_1 - x_2)^{-m-1} - (-x_2 + x_1)^{-m-1}).$$

Since $(x_1 - x_2)^r = (-x_2 + x_1)^r$ for $r \in \mathbb{N}$, we find that

$$\sum_{m\in\mathbb{Z}} x_0^m((x_1 - x_2)^{-m-1} - (-x_2 + x_1)^{-m-1}) = \sum_{m\in\mathbb{N}} x_0^m((x_1 - x_2)^{-m-1} - (-x_2 + x_1)^{-m-1}).$$

Finally an application of Lemma 2.2.4 and Theorem 2.1.5 gives

$$\begin{aligned}\sum_{m\in\mathbb{N}} x_0^m((x_1 - x_2)^{-m-1} - (-x_2 + x_1)^{-m-1}) &= \sum_{m\in\mathbb{N}} \frac{1}{m!}\left(-x_0\frac{\partial}{\partial x_1}\right)^m x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) \\ &= x_2^{-1}e^{-x_0\frac{\partial}{\partial x_1}}\delta\left(\frac{x_1}{x_2}\right) \\ &= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right),\end{aligned}$$

as desired. □

2.3 Vertex algebras and vertex operator algebras

A *vertex algebra* is a quadruple $(V, Y, \mathbb{1}, d)$ where V is a vector space,

$$\begin{aligned} Y(\cdot, x) : V &\longrightarrow (\mathrm{End}\, V)[[x, x^{-1}]] \\ a &\mapsto \sum_{n\in\mathbb{Z}} a_n x^{-n-1} \end{aligned}$$

is a linear map, $\mathbb{1}$ is a distinguished vector, and $d : V \to V$ is an endomorphism of V so that the following conditions are satisfied

(VA1) $Y(\mathbb{1}, x) = \mathrm{id}_V$;

(VA2) $Y(a, x)\mathbb{1} \in V[[x]]$ and $\lim_{x \to 0} Y(a, x)\mathbb{1} = a$ for $a \in V$;

(VA3) $[d, Y(a, x)] = Y(da, x) = \frac{d}{dx} Y(a, x)$ for any $a \in V$;

(VA4) $Y(a, x)b \in V((x))$ for any $a, b \in V$;

(VA5) The *Jacobi identity* holds:

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(a, x_1)Y(b, x_2)c - x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y(b, x_2)Y(a, x_1)c$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(a, x_0)b, x_2)c$$

for any $a, b, c \in V$.

If $(V_1, Y_1, \mathbb{1}^1, d_1)$ and $(V_2, Y_2, \mathbb{1}^2, d_2)$ are vertex algebras, then a *vertex algebra homomorphism* from V_1 into V_2 is linear map $f : V_1 \to V_2$ such that

(Hom1) f is compatible with Y_1 and Y_2 in the sense that

$$Y_2(f(u), x)f(v) = f(Y_1(u, x)v) \quad \text{for each } u, v \in V_1,$$

or equivalently,

$$f(u)_n f(v) = f(u_n v) \quad \text{for each } u, v \in V_1 \text{ and } n \in \mathbb{Z};$$

(Hom2) f sends $\mathbb{1}^1$ to $\mathbb{1}^2$;

(Hom3) f is compatible with d_1 and d_2 in the sense that $f \circ d_1 = d_2 \circ f$.

A vertex algebra $(V, Y, \mathbb{1}, d)$ is called a *vertex operator algebra* if there is another distinguished vector $\omega \in V$ called the *Virasoro vector* such that

(VOA1) the vertex operators associated with ω form a representation of the Virasoro algebra, that is, if we set $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}$, then

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12}\delta_{m+n,0}c_V$$

for some $c_V \in \mathbb{C}$ call the *central charge of* V;

(VOA2) $L(-1) \equiv d \in (\mathrm{End}\, V)$;

(VOA3) V is $\mathbb{Z}$-graded by the finite-dimensional eigenspaces for $L(0)$ which are truncated from below, that is, we may write

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}$$

where $V_{(n)} = \{a \in V : L(0)v = na\}$ and we have $\dim V_{(n)} < \infty$ and $V_{(n)} = 0$ for n sufficiently negative.

In this case, we would write that the quadruple $(V, Y, \mathbb{1}, \omega)$ is a vertex operator algebra. Note that there is no need to specify the endomorphism d anymore, since by axiom (VOA2), we know that $d = L(-1) = \omega_0$.

If $(V_1, Y_1, \mathbb{1}^1, \omega^1)$ and $(V_2, Y_2, \mathbb{1}^2, \omega^2)$ are vertex operator algebras, a *vertex operator algebra homomorphism* from V_1 into V_2 is a vertex algebra homomorphism $f : V_1 \to V_2$ such that $f(\omega_1) = \omega_2$. Note that this implies in particular that the central charges of the vertex operator algebras V_1 and V_2 must be the same.

The category of vertex operator algebras of central charge c, denoted $\mathbf{VOA}(c)$, has as its objects vertex operator algebras of central charge c and as its morphisms vertex operator algebra homomorphisms between these objects. More generally, the category of vertex operator algebras, denoted **VOA**, has as its objects vertex operator algebras and as its morphisms vertex operator algebra homomorphisms.

A *module* for a vertex algebra $(V, Y, \mathbb{1}, d)$ is a pair (M, Y_M, D) where M is a vector space,

$$Y(\cdot, x) : V \longrightarrow (\mathrm{End}\, M)[[x, x^{-1}]]$$

is a linear map, and $D \in \mathrm{End}\, M$ is an endomorphism of M so that the following conditions are satisfied

(M1) $Y_M(\mathbb{1}, x) = \mathrm{id}_M$;

(M2) $Y_M(a, x)u \in M((x))$ for any $a \in V$ and $u \in M$;

(M3) $[D, Y_M(a, x)] = Y_M(da, x) = \frac{d}{dx} Y_M(a, x)$ for $a \in V$;

(M4) For any $a, b \in V$ and $u \in M$, the modified Jacobi identity holds

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_M(a, x_1) Y_M(b, x_2) u - x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y_M(b, x_2) Y_M(a, x_1) u$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_M(Y(a, x_0)b, x_2) u.$$

In particular, if $(V, Y, \mathbb{1}, d)$ is a vertex algebra, then (V, Y_V, d) is a module for itself if we set $Y_V \equiv Y$.

If (M_1, Y_1, D_1) and (M_2, Y_2, D_2) are two modules for a vertex algebra $(V, Y, \mathbb{1}, d)$, a *homomorphism of V-modules from M_1 into M_2* is a linear map $f : M_1 \to M_2$ which is compatible with the actions of Y_1 and Y_2 in the sense that

$$f(Y_1(a,x)u) = Y_2(a,x)f(u)$$

for each $a \in V$ and $u \in M_1$.

Now let $(V, Y, \mathbb{1}, \omega)$ be a vertex operator algebra. A *weak module* for the vertex operator algebra V is a module for the vertex algebra $(V, Y, \mathbb{1}, L(-1))$. If (M_1, Y_1, D_1) and (M_2, Y_2, D_2) are weak modules for V, a *weak V-module morphism* from M_1 into M_2 is a linear map $f : M_1 \to M_2$ that is compatible with the action of Y in the sense that

$$f(a_n(v)) = a_n(f(v)) \quad \text{for each } a \in V \text{ and } v \in M_1. \tag{2.3.1}$$

The category of weak V-modules is denoted $\mathbf{Weak}(V)$.

There is a slightly stronger notion of a module which enjoys an $\mathbb{N}$-grading that is compatible with the $\mathbb{Z}$-grading of V. In particular, a weak module (M, Y_M, D) is called a *weak-admissible module* for the vertex operator algebra V if M carries an $\mathbb{N}$-grading $M = \bigoplus_{n \in \mathbb{N}} M(n)$ such that

$$a_k(M(j)) \subset M(j + n - k - 1) \quad \text{for homogeneous } a \in V_{(n)}.$$

A morphism of two weak admissible V-modules M_1 and M_2 is a morphism of weak V-modules $f : M_1 \to M_2$. The category of weak-admissible V-modules is denoted $\mathbf{Adm}(V)$.

There is yet a stronger notion of a module where the grading of the module is compatible with the eigenspaces for the vertex operator $\omega_1 = L(0)$. More specifically, an *ordinary module* for the vertex operator algebra V is a weak V-module M which carries a $\mathbb{C}$-grading $M = \coprod_{\lambda \in \mathbb{C}} M_\lambda$ such that

(OM1) The dimension $\dim M_\lambda$ is finite,

(OM2) For fixed λ, we have $M_{\lambda+n} = 0$ for $n \in \mathbb{Z}$ small enough,

(OM3) The eigenspaces for the operator $L(0) = \omega_1 \in \operatorname{End} M$ give M its grading, in the sense that

$$M_\lambda = \{u \in M : L(0)u = \lambda u\}.$$

A morphism of ordinary modules is simply a morphism of weak modules. This implies in particular that morphisms respect the $\mathbb{C}$-gradings (see Exercise 2.3.3). The category of ordinary modules for a vertex operator algebra V is denoted $\mathbf{Ord}(V)$.

As an example of a trivial vertex algebra, consider the quadruple

$$E = (\mathbb{C}, Y_E, 1, d_E) \tag{2.3.2}$$

where $Y_E(\cdot, x) : \mathbb{C} \to (\operatorname{End}\mathbb{C})[[x, x^{-1}]]$ is the unique linear map satisfying $Y_E(1, x) = \mathrm{id}_{\mathbb{C}}$ and where $d_E : \mathbb{C} \to \mathbb{C}$ is the zero map. Then the quadruple E so defined is a vertex algebra, albeit a trivial one. Indeed, the axioms (VA1) through (VA4) are easily verified, and axiom (VA5) uses the three-term delta function identity 2.2.2.

Moreover, we may consider the vertex algebra E to be a vertex operator algebra if we specify the Virasoro vector to be $\omega = 0$. The vector space $\mathbb{C}$ is then trivially graded, with the whole vector space $\mathbb{C}$ lying in the weight zero eigenspace for $L(0) = \omega_{-1}$. The central charge of this vertex operator algebra is trivial $c_E = 0$.

At this point, we could give more examples of vertex operator algebras. However, to verify that they indeed satisfy the axioms above, we would require more tools and a much more extensive study. We direct the reader to [L96, LL, Ba13] for more discussion of the axioms of vertex operator algebras and for some nontrivial examples.

Exercise 2.3.1. Let $(V, Y, \mathbb{1}, \omega)$ be a vertex operator algebra.

(a) Show that the vacuum vector is unique.

(b) Show that for each $v \in V$, we have $v_{-1}\mathbb{1} = v$.

(c) Show that the map $Y(\cdot, x) : V \to (\operatorname{End} V)[[x, x^{-1}]]$ is injective.

Exercise 2.3.2. Show that for each vectors u and v in a vertex operator algebra V, we have

$$[Y(u, x_1), Y(v, x_2)] = \operatorname{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2),$$

where $\operatorname{Res}_{x_0}(A)$ is the coefficient of x_0^{-1} in the element A and where $[-, -]$ denotes the commutator.

Exercise 2.3.3. Let $(V, Y, \mathbb{1}, \omega)$ be a vertex operator algebra. Let (M_1, Y_1) and (M_2, Y_2) be ordinary modules for V and let $f : M_1 \to M_2$ be a morphism of ordinary V-modules. Show that f preserves the $\mathbb{C}$-grading of M_1 and M_2.

2.4 Example: The space of one free boson

In this section, we give an example of a non-trivial vertex operator algebra, which, following for instance [Ba13], we call the *space of one free boson*. This vertex operator algebra will be formulated as a local system of vertex operators acting on some vector space (cf. [L96]).

Recall that a vertex operator algebra is foremost a vector space V. In the case of one free boson, we set our vector space to be

$$V_{bos} = \mathbb{C}[a_1, a_2, \ldots],$$

that is, the space of polynomials in an infinite number of formal commuting variables. Note that, on the one hand, this space has the structure of a commutative $\mathbb{C}$-algebra. We will show that this space can also be endowed with the structure of a nontrivial vertex algebra.

Next, we are to select a linear map $Y_{bos}(\cdot, x) : V_{bos} \to (\mathrm{End}\, V_{bos})[[x, x^{-1}]]$. However, this is by far the most complicated of the data from the quadruple $(V, Y, \mathbb{1}^{bos}, \omega^{bos})$, so we save the task of defining $Y_{bos}(\cdot, x)$ for last.

We are also called to distinguish two vectors $\mathbb{1}^{bos}$ and ω^{bos}. These vectors are supposed to be homogeneous of weight zero and two respectively (in the $\mathbb{Z}$-grading given by the operator $L(0) = \omega_1$.) Selecting ω^{bos} seems kind of circular, since a selection of ω determines $L(0)$, which determines a grading, but conversely, a selection of a $\mathbb{Z}$-grading limits our selection of homogeneous vectors of weight two. Nevertheless, we will choose to select a natural $\mathbb{Z}$-grading on V_{bos} first, before discussing how to form ω.

The space V_{bos} enjoys a natural $\mathbb{Z}$-grading. Indeed, it would seem appropriate to declare a_k to be homogeneous of weight k, and more generally, to let the monomial $a_{i_1}^{r_1} \cdots a_{i_s}^{r_s}$ to be homogeneous of weight $\sum_{j=1}^{s} r_j i_j$. In this way, we obtain a $\mathbb{Z}$-grading of the algebra V_{bos}.

The vector $\mathbb{1}^{bos}$ is supposed to be homogeneous of weight zero. By our grading above, this means that $\mathbb{1}^{bos}$ is supposed to be some constant polynomial. It would seem natural to select the particular constant polynomial given by $1 \in \mathbb{C}$ so let us do that, and set $\mathbb{1}^{bos} = 1 \in V_{bos}$.

Now the vector ω^{bos} is supposed to be homogeneous of weight two in V_{bos}. Note that this implies ω^{bos} belongs to the subspace $\mathbb{C}a_1^2 \oplus \mathbb{C}a_2$. It turns out that $\omega^{bos} = \frac{1}{2}a_1^2$ is a good choice.

It is finally time to determine the action of Y_{bos} on our space V_{bos}. We will do this in steps. The idea will be to determine the action of Y_{bos} on the variables a_k inductively and then on the monomials using a sort of product rule which is called *normal ordering*. We then of course extend the action linearly to our whole space V_{bos}.

To satisfy axiom (VA1), we set $Y_{bos}(\mathbb{1}^{bos}, x) = 1$. We then set $Y_{bos}(a_1, x) = \sum_{n \in \mathbb{Z}} (a_1)_n x^{-n-1}$ where

$$(a_1)_n = \begin{cases} a_{-n} & n < 0 \\ n\frac{\partial}{\partial a_n} & n > 0 \\ 0 & n = 0 \end{cases}.$$

The operator a_j is to be understood as left multiplication by the formal variable a_j in the commutative algebra structure of V_{bos}, which is clearly an endomorphism of V_{bos}. The operator $\frac{\partial}{\partial a_j}$ is the partial derivative endomorphism defined in Section 2.1.

We now define the action of $Y(\cdot, x)$ on a_k for $k > 1$. Inductively, we set

$$Y_{bos}(a_k, x) = \frac{1}{(k-1)!}\left(\frac{\partial}{\partial x}\right)^{k-1} Y_{bos}(a_1, x)$$

for $k \geq 2$. If we write

$$Y_{bos}(a_k, x) = \sum_{n\in\mathbb{Z}} (a_k)_n x^{-n-1}$$

then it is easy to see that each $(a_k)_n$ is an endomorphism of V_{bos}.

To define our operator $Y_{bos}(\cdot, x)$ on products of variables, we need to introduce the notion of *normal ordering*, which is a type of product rule for Y_{bos}. We do this for general $u, v \in V_{bos}$. If $u \in V_{bos}$, we write

$$Y^{+}_{bos}(u, x) = \sum_{n<0} u_n x^{-n-1}$$
$$Y^{-}_{bos}(u, x) = \sum_{n\geq 0} u_n x^{-n-1}$$

and we call $Y^{+}_{bos}(u, x)$ and $Y^{-}_{bos}(u, x)$ the *regular* and *singular* parts of $Y_{bos}(u, x)$ respectively. We then set

$${}^{\circ}_{\circ}Y(u, x_1)Y(v, x_2){}^{\circ}_{\circ} = Y^{+}(u, x_1)Y(v, x_2) + Y(v, x_2)Y^{-}(u, x_1)$$

and we call ${}^{\circ}_{\circ}Y(u, x_1)Y(v, x_2){}^{\circ}_{\circ}$ the *normal-ordered* product of $Y_{bos}(u, x_1)$ and $Y_{bos}(v, x_2)$.

Now if $u, v \in V_{bos}$, we set

$$Y_{bos}(uv, x) = {}^{\circ}_{\circ}Y_{bos}(u, x)Y_{bos}(v, x){}^{\circ}_{\circ}. \tag{2.4.1}$$

Note that we are able to multiply u and v in the left hand side of (2.4.1) since V_{bos} also carries the structure of a commutative $\mathbb{C}$-algebra (recall $V_{bos} = \mathbb{C}[a_1, a_2, \ldots]$). Using (2.4.1), we are now able to define $Y(v, x)$ for any $v \in V_{bos}$.

The claim is that with these definitions the quadruple $(V_{bos}, Y_{bos}, \mathbb{1}^{bos}, \omega^{bos})$ acquires the structure of a vertex operator algebra. Of course, there are many things to show in order to prove this claim, and we outline a few of the steps in the exercises. However, a major development in the theory of vertex operator algebras was a simplification of the axioms down to a much more manageable set of axioms (cf. [L96]). These axioms are also discussed in the exercises.

Exercise 2.4.1. Show that V_{bos} satisfies the *truncation condition* (VA4) for elements a of the form $a_1, a_2, \ldots$. Can you reason why the truncation condition holds for all $a \in V_{bos}$?

Exercise 2.4.2. Show that the *creation property* (VA2) is satisfied for a of the form $a_1, a_2, \ldots$ in V_{bos}. Can you reason why the creation property holds for all $a \in V_{bos}$?

Exercise 2.4.3. Set $Y_{bos}(\omega^{bos}, x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$.

(a) Find the operators $L(n) : V_{bos} \to V_{bos}$ for $n = -1, 0, 1, 2, 3$.

(b) Show that the operator $L(0)$ grades the space V_{bos} in the sense that if $a \in V_{(n)}$, then $L(0)a = na$.

(c) Given that $L(2)\omega^{bos} = \frac{1}{2}c_{bos}\mathbb{1}^{bos}$, calculate the central charge c_{bos} for V_{bos}.

(d) One can show (cf. [L96]) that for the $L(-1)$-derivative property to hold on V_{bos} (given the way we defined Y_{bos} and ω^{bos}), it suffices to show that

$$Y_{bos}(L(-1)a_1, x) = \frac{d}{dx}Y_{bos}(a_1, x).$$

Show that this sufficient condition holds.

Exercise 2.4.4. (a) Show that the commutator

$$[Y_{bos}(a_1, x_1), Y_{bos}(a_2, x_2)]$$

is nonzero in $(\mathrm{End}\, V_{bos})[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$.

(b) One can show (cf. [L96]) that for the Jacobi identity to hold on V_{bos} (given the way we defined Y_{bos}), it suffices to show that there is an integer k so that

$$(x_1 - x_2)^k[Y_{bos}(a_1, x_1), Y_{bos}(a_1, x_2)] = 0.$$

Find such a k. We remark that this exercise shows that although the elements $Y_{bos}(a_1, x_1)$ and $Y_{bos}(a_1, x_2)$ do not commute in $(\mathrm{End}\, V_{bos})[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$, they do commute up to a clearing of a formal pole.

Chapter 3

Vertex operator algebras as a symmetric monoidal category

3.1 Basic category theory

A *category* $\mathcal{C}$ consists of the following data:

- A set $\mathrm{Obj}(\mathcal{C})$ of objects;
- For each pair of objects $X, Y \in \mathrm{Obj}(\mathcal{C})$, a collection $\mathcal{C}(X, Y)$ of *morphisms* from X to Y;
- For objects $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, a function $\circ : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$, which assigns to a pair of morphisms $(g, f) \in \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y)$ their *composite morphism* $g \circ f \in \mathcal{C}(X, Z)$;
- For each object $X \in \mathrm{Obj}(\mathcal{C})$, an element $\mathrm{id}_X \in \mathcal{C}(X, X)$ called the *identity morphism* of X

subject to the following rules:

- *Composition is associative:* For objects $W, X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, if $f \in \mathcal{C}(X, Y), g \in \mathcal{C}(Y, Z), h \in \mathcal{C}(Z, W)$, then $(h \circ g) \circ f = h \circ (g \circ f)$;
- The *unit laws* are satisfied: If $f \in \mathcal{C}(X, Y)$, then $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$.

We typically denote the collection of morphisms of a category $\mathcal{C}$ by

$$\mathrm{Mor}(\mathcal{C}) = \bigcup_{X, Y \in \mathcal{C}} \mathcal{C}(X, Y).$$

Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$. The morphism f is called an *isomorphism* if there is another morphism $f^{-1} : Y \to X$ such that $f \circ f^{-1} = \mathrm{id}_Y$ and $f^{-1} \circ f = \mathrm{id}_X$. It is easily shown that if such a morphism f^{-1} exists, then it is unique and called the *inverse* of f.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $\mathcal{F}$ from $\mathcal{C}$ into $\mathcal{D}$ is a map which assigns to each object $X \in \mathrm{Obj}(\mathcal{C})$ an object $\mathcal{F}(X) \in \mathrm{Obj}(\mathcal{D})$ and to each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ a morphism $\mathcal{F}(f) \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ such that the following properties are satisfied:

- *Compatibility with composition*: $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ whenever $g \circ f$ is defined;
- *Compatibility with identities*: For each $X \in \mathrm{Obj}(\mathcal{C})$, we have $\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$.

If $\mathcal{F}$ and $\mathcal{G}$ are two functors from $\mathcal{C}$ into $\mathcal{D}$, then a *natural transformation* from $\mathcal{F}$ into $\mathcal{G}$ is an assignment

$$\begin{aligned} \eta : \mathcal{C} &\longrightarrow \mathrm{Mor}(\mathcal{D}) \\ X &\mapsto \eta_X \in \mathcal{D}(\mathcal{F}(X), \mathcal{G}(X)) \end{aligned}$$

to each object $X \in \mathcal{C}$ a morphism $\eta_X : \mathcal{F}(X) \to \mathcal{G}(X)$ such that the following property holds for each pair of objects X and Y in $\mathcal{C}$: for each morphism $f \in \mathcal{C}(X, Y)$, the diagram

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ \mathcal{G}(X) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(Y) \end{array}$$

commutes. If for each object $X \in \mathcal{C}$, the morphism η_X is an isomorphism, then we say that η is a *natural isomorphism* from $\mathcal{F}$ into $\mathcal{G}$ and the functors $\mathcal{F}$ and $\mathcal{G}$ are called *naturally isomorphic*.

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We form a new category $\mathcal{C} \times \mathcal{D}$ called the *product category* in the following manner:

- The objects of $\mathcal{C} \times \mathcal{D}$ are pairs (X, Y) where X is an object of $\mathcal{C}$ and Y is an object of $\mathcal{D}$;
- A morphism from (X_1, Y_1) to (X_2, Y_2) is a pair (f, g) where $f \in \mathcal{C}(X_1, X_2)$ and $g \in \mathcal{D}(Y_1, Y_2)$;
- The composition of $\mathcal{C} \times \mathcal{D}$ is the composition induced by the point-wise composition composition from $\mathcal{C}$ and $\mathcal{D}$:

$$(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1);$$

- For an object (X, Y) of $\mathcal{C} \times \mathcal{D}$, the corresponding identity morphism is $(\mathrm{id}_X, \mathrm{id}_Y)$.

It is routine to check that this construction indeed defines a new category.

Let $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{D}$ be categories and let $\mathcal{C}_1 \times \mathcal{C}_2$ denote the product category of $\mathcal{C}_1$ and $\mathcal{C}_2$. A *bifunctor* from $\mathcal{C}_1$ and $\mathcal{C}_2$ into $\mathcal{D}$ is a functor

$$\mathcal{F} : \mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow \mathcal{D}$$

from the product category $\mathcal{C}_1 \times \mathcal{C}_2$ into $\mathcal{D}$.

More generally, a *multifunctor* is a functor out of a multiproduct category into some other category.

Exercise 3.1.1. Show that the product category construction indeed defines a new category.

3.2 Monoidal categories

For a category $\mathcal{C}$ with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, we define the following functors. For objects X, Y, Z and morphisms f, g, h of $\mathcal{C}$, we have:

- The identity functor $\mathcal{I} : \mathcal{C} \to \mathcal{C}$ defined by

$$\begin{aligned} X &\mapsto X \\ f &\mapsto f; \end{aligned}$$

- The bifunctor $\otimes^{op} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ defined by

$$\begin{aligned} (X, Y) &\mapsto Y \otimes X \\ (f, g) &\mapsto g \otimes f; \end{aligned}$$

- The tri-functor $\mathcal{A}_L^{\otimes} : (\mathcal{C} \times \mathcal{C}) \times \mathcal{C} \to \mathcal{C}$ defined by

$$\begin{aligned} (X, Y, Z) &\mapsto (X \otimes Y) \otimes Z \\ (f, g, h) &\mapsto (f \otimes g) \otimes h; \end{aligned}$$

- The tri-functor $\mathcal{A}_R^{\otimes} : \mathcal{C} \times (\mathcal{C} \times \mathcal{C}) \to \mathcal{C}$ defined by

$$\begin{aligned} (X, Y, Z) &\mapsto X \otimes (Y \otimes Z) \\ (f, g, h) &\mapsto f \otimes (g \otimes h); \end{aligned}$$

- The functor $\mathcal{L}_X^{\otimes} : \mathcal{C} \to \mathcal{C}$ defined by

$$\begin{aligned} Y &\mapsto X \otimes Y \\ f &\mapsto \mathrm{id}_X \otimes f; \end{aligned}$$

- The functor $\mathcal{R}_X^{\otimes} : \mathcal{C} \to \mathcal{C}$ defined by

$$\begin{aligned} Y &\mapsto Y \otimes X \\ f &\mapsto f \otimes \mathrm{id}_X . \end{aligned}$$

A *monoidal category* is a six-tuple $(\mathcal{C}, \otimes, E, \alpha, \lambda, \rho)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, E is a distinguished object of $\mathcal{C}$, and α, λ, ρ are natural isomorphisms of functors so that the following conditions are satisfied:

(MC1) The map α is a natural isomorphism of $\mathcal{A}_L^{\otimes}$ into $\mathcal{A}_R^{\otimes}$;

(MC2) The map λ is a natural isomorphism of $\mathcal{L}_E^{\otimes}$ into $\mathcal{I}$;

(MC3) The map ρ is a natural isomorphism of $\mathcal{R}_E^{\otimes}$ into $\mathcal{I}$;

(MC4) For objects W, X, Y, Z of $\mathcal{C}$, the following *pentagon diagram* commutes:

$$\begin{array}{ccccc} & & (W \otimes X) \otimes (Y \otimes Z) & & \\ & \overset{\alpha_{W\otimes X, Y, Z}}{\nearrow} & & \overset{\alpha_{W, X, Y\otimes Z}}{\searrow} & \\ ((W \otimes X) \otimes Y) \otimes Z & & & & W \otimes (X \otimes (Y \otimes Z)) \\ \downarrow {\scriptstyle \alpha_{W,X,Y} \otimes \mathrm{id}_Z} & & & & \downarrow {\scriptstyle \mathrm{id}_W \otimes \alpha_{X,Y,Z}} \\ (W \otimes (X \otimes Y)) \otimes Z & & \xrightarrow{\alpha_{W, X\otimes Y, Z}} & & W \otimes ((X \otimes Y) \otimes Z) \end{array}$$

(MC5) For objects X, Y in $\mathcal{C}$, the following *triangle diagram* commutes:

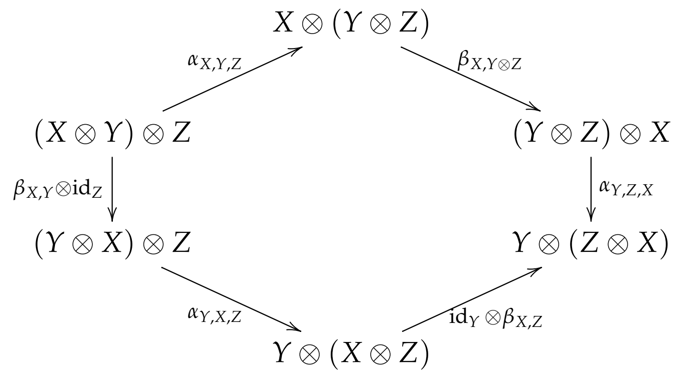
$$\begin{array}{ccc} (X \otimes E) \otimes Y & \xrightarrow{\alpha_{X,E,Y}} & X \otimes (E \otimes Y) \\ & {\scriptstyle \rho_X \otimes \mathrm{id}_Y} \searrow & \downarrow {\scriptstyle \mathrm{id}_X \otimes \lambda_Y} \\ & & X \otimes Y \end{array}$$

A *braided monoidal category* is a seven-tuple $(\mathcal{C}, \otimes, E, \alpha, \lambda, \rho, \beta)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, E is a distinguished object of $\mathcal{C}$, and $\alpha, \lambda, \rho, \beta$ are natural isomorphisms of functors so that the following conditions are satisfied:

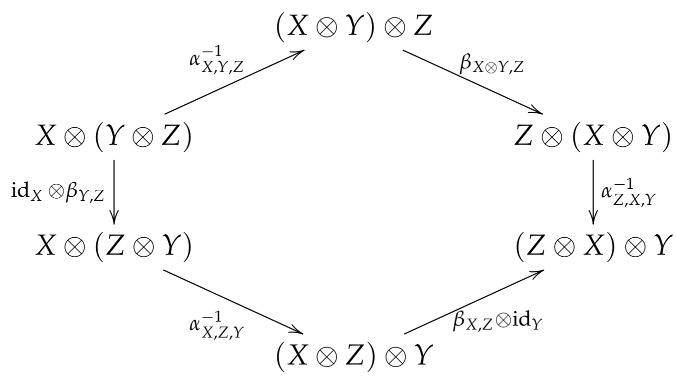
(BCM1) The six-tuple $(\mathcal{C}, \otimes, E, \alpha, \lambda, \rho)$ is a monoidal category;

(BCM2) The map β is a natural isomorphism (called the *braiding*) of $\otimes$ into $\otimes^{op}$;

(BCM3) For objects X, Y, Z of $\mathcal{C}$, the following *first hexagon diagram* commutes:



(BCM4) For objects X, Y, Z of $\mathcal{C}$, the following *second hexagon diagram* commutes:



A *symmetric monoidal category* is a seven-tuple $(\mathcal{C}, \otimes, E, \alpha, \lambda, \rho, \beta)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, E is a distinguished object of $\mathcal{C}$, and $\alpha, \lambda, \rho, \beta$ are natural isomorphisms of functors so that the following conditions are satisfied:

(SMC1) The seven-tuple $(\mathcal{C}, \otimes, E, \alpha, \lambda, \rho, \beta)$ is a braided monoidal category;

(SMC2) The *braiding isomorphism* β satisfies the property

$$\beta_{Y,X} \circ \beta_{X,Y} = \mathrm{id}_{X \otimes Y} .$$

3.3 Monoidal functors

Let $(\mathcal{C}, \otimes_{\mathcal{C}}, E_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, E_{\mathcal{D}}, \alpha^{\mathcal{D}}, \lambda^{\mathcal{D}}, \rho^{\mathcal{D}})$ be monoidal categories. A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is called *lax monoidal* if there is a morphism $\epsilon \in \mathrm{Hom}_{\mathcal{D}}(E_{\mathcal{D}}, \mathcal{F}(E_{\mathcal{C}}))$ and a natural transformation

$$\begin{aligned}\phi : \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{C}) &\longrightarrow \mathrm{Mor}(\mathcal{D})\\ (X,Y) &\mapsto \phi_{X,Y} \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X) \otimes_{\mathcal{D}} \mathcal{F}(Y), \mathcal{F}(X \otimes_{\mathcal{C}} Y))\end{aligned}$$

such that the following conditions are satisfied for objects X, Y, Z of $\mathcal{C}$:

(LM1) Associativity holds:

$$\begin{array}{ccc} \mathcal{F}(X) \otimes \mathcal{F}(Y) \otimes \mathcal{F}(Z) & \xrightarrow{\mathrm{id}_{\mathcal{F}(X)} \otimes \phi_{Y,Z}} & \mathcal{F}(X) \otimes \mathcal{F}(Y \otimes Z) \\ \downarrow{\scriptstyle \phi_{X,Y} \otimes \mathrm{id}_{\mathcal{F}(Z)}} & & \downarrow{\scriptstyle \phi_{X,Y\otimes Z}} \\ \mathcal{F}(X \otimes Y) \otimes \mathcal{F}(Z) & \xrightarrow{\phi_{X\otimes Y,Z}} & F(X \otimes Y \otimes Z) \end{array}$$

(LM2) We have

$$\begin{array}{ccc} E_{\mathcal{D}} \otimes \mathcal{F}(X) & \xrightarrow{\epsilon \otimes \mathrm{id}_{\mathcal{F}(X)}} & \mathcal{F}(E_{\mathcal{D}}) \otimes \mathcal{F}(X) \\ \downarrow{\scriptstyle \lambda^{\mathcal{D}}_{\mathcal{F}(X)}} & & \downarrow{\scriptstyle \phi_{E_{\mathcal{C}},X}} \\ \mathcal{F}(X) & \xleftarrow{\mathcal{F}(\lambda^{\mathcal{C}}_X)} & \mathcal{F}(E_{\mathcal{C}} \otimes X) \end{array}$$

(LM3) We have

$$\begin{array}{ccc} \mathcal{F}(X) \otimes E_{\mathcal{D}} & \xrightarrow{\mathrm{id}_{\mathcal{F}(X)} \otimes \epsilon} & \mathcal{F}(X) \otimes \mathcal{F}(E_{\mathcal{D}}) \\ \downarrow{\scriptstyle \rho^{\mathcal{D}}_{\mathcal{F}(X)}} & & \downarrow{\scriptstyle \phi_{X,E_{\mathcal{C}}}} \\ \mathcal{F}(X) & \xleftarrow{\mathcal{F}(\rho^{\mathcal{C}}_X)} & \mathcal{F}(X \otimes E_{\mathcal{C}}) \end{array}$$

A lax monoidal functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is called a *braided monoidal functor* if for each pair of objects X, Y in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(X) \otimes \mathcal{F}(Y) & \xrightarrow{\beta_{\mathcal{F}(X),\mathcal{F}(Y)}} & \mathcal{F}(Y) \otimes \mathcal{F}(X) \\ \downarrow{\scriptstyle \phi_{X,Y}} & & \downarrow{\scriptstyle \phi_{Y,X}} \\ \mathcal{F}(X \otimes Y) & \xrightarrow{\mathcal{F}(\beta_{X,Y})} & \mathcal{F}(Y \otimes X) \end{array} \tag{BMF}$$

If moreover, the categories $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal categories, then $\mathcal{F}$ is called a *symmetric monoidal functor*.

3.4 Ring modules

Let R be a ring with identity 1_R. The category $\mathbf{Mod}_R$ has as its objects left R-modules and as its morphisms module homomorphisms. This category comes equipped with a bifunctor $\otimes$ given by the tensor product, which we describe now.

Let us first recall that a (left) R-module is an additive group M together with an action

$$\begin{aligned} R \times M &\longrightarrow M \\ (r, m) &\mapsto r \cdot m \end{aligned}$$

of R on M which is compatible with the structures on M and R in the sense that

(i) $1_R \cdot m = m$ for each $m \in M$;

(ii) $(r + s) \cdot m = r \cdot m + s \cdot m$ for each $r, s \in R$ and $m \in M$;

(iii) $(rs) \cdot m = r \cdot s \cdot m$ for each $r, s \in R$ and $m \in M$;

(iv) $r \cdot (m + m') = r \cdot m + r \cdot m'$ for each $r \in R$ and $m, m' \in M$.

If M and N are two R-modules, then an R-module homomorphism from M into N is a group homomorphism $f : M \to N$ which is compatible with the action of R in the sense that

$$r \cdot f(m) = f(r \cdot m)$$

for each $r \in R$ and $m \in M$.

If M and N are two R-modules, then we may form another R-module $M \times N$ called the *direct sum of M in N* in the following manner. We let $M \times N$ denote the direct product of M and N (as groups) and we let R act on $M \times N$ by setting

$$R \times (M \times N) \longrightarrow M \times N \tag{3.4.1}$$
$$(r, (m, n)) \mapsto (r \cdot m, r \cdot n). \tag{3.4.2}$$

It is routine to show that $M \times N$ is an R-module under this action.

Let M, N, P be R-modules. A *bilinear map* $\phi : M \times N \to P$ is a function such that

$$\begin{aligned} \phi(r \cdot m + m', n) &= r \cdot \phi(m, n) + \phi(m', n) \\ \phi(m, r \cdot n + n') &= r \cdot \phi(m, n) + \phi(m, n') \end{aligned}$$

for each $r \in R$, each $m, m' \in M$ and each $n, n' \in N$.

Let M and N be two left R-modules. Then a *tensor product* of M and N is pair $(M \otimes_R N, \phi)$ where $M \otimes_R N$ is a left R-module and $\phi : M \times N \to M \otimes_R N$ is a bilinear map such that the following *universal property holds:* For each bilinear map $B : M \times N \to P$ into a left R-module P, there is a unique linear map $b : M \otimes_R N \to P$ such that $b \circ \phi = B$.

The universal property actually guarantees the if a tensor product exists, then it is unique up to isomorphism. That is, we have the following result, whose proof we leave as an exercise.

Lemma 3.4.1. *Let M and N be two R-modules. If (T, ϕ_T) and $(T', \phi_{T'})$ are two tensor products of M and N, then T and T' are isomorphic as R-modules.*

For a set X, we let $\mathcal{F}_R(X)$ denote the free R-module generated by X. This means that $\mathcal{F}_R(X)$ is the set of all symbols of the form

$$r_1 x_1 + \cdots + r_k x_k$$

for some $k \geq 1$, for $r_i \in R$ and $x_i \in X$. The set $\mathcal{F}_R(X)$ is an additive group by the rule

$$\sum_{i \in I} r_i x_i + \sum_{i \in I} s_i x_i = \sum_{i \in I} (r_i + s_i) x_i.$$

The action of R on $\mathcal{F}_R(X)$ is defined by

$$r \cdot (r_1 x_1 + \cdots + r_k x_k) = (r r_1) x_1 + \cdots + (r r_k) x_k.$$

It is a simple exercise to show that $\mathcal{F}_R(X)$ is indeed an R-module.

If X is a set and $\mathcal{F}_R(X)$ denotes the free R-module generated by X, then there is a natural inclusion map $\iota : X \to \mathcal{F}_R(X)$ defined by sending $x_i \mapsto 1_R x_i$.

We now construct a tensor product of R-modules. We omit, however, a proof of the theorem and refer the reader to [DF] for further discussion.

Theorem 3.4.2. *Let M and N be two R-modules. Let $\mathcal{F}_R(M \times N)$ denote the free module generated by the cartesian product $M \times N$ and let $\iota : M \times N \to \mathcal{F}_R(M \times N)$ denote the natural inclusion map. Let I denote the submodule of $\mathcal{F}_R(M \times N)$ generated by all elements of the form*

$$\begin{aligned} (m + m', n) - (m, n) - (m', n) \\ (m, n + n') - (m, n) - (m, n') \\ r \cdot (m, n) - (r \cdot m, n) \\ r \cdot (m, n) - (m, r \cdot n), \end{aligned}$$

and let $\pi : \mathcal{F}_R(M \times N) \to \mathcal{F}_R(M \times N)/I$ denote the projection map. Let $M \otimes_R N$ denote the quotient module $\mathcal{F}_R(M \times N)/I$ and let $\phi : M \times N \to M \otimes_R N$ denote the composition $\phi = \pi \circ \iota$. Then the pair $(M \otimes_R N, \phi)$ is a tensor product of M and N.

The construction $M \otimes_R N$ is usually called *the* tensor product of M and N, but it is really only unique up to isomorphism.

Note that the construction $\otimes_R$ above describes a mapping of objects needed to define a bifunctor

$$\otimes_R : \mathbf{Mod}_R \times \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_R.$$

We now extend $\otimes_R$ to a bifunctor by describing its action on morphisms.

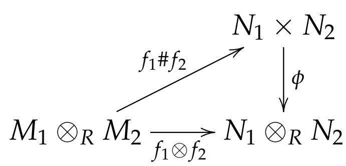
If $f_1 : M_1 \to N_1$ and $f_2 : M_2 \to N_2$ are R-module homomorphisms, then we may form a bilinear map $f_1 \# f_2 : M_1 \times M_2 \to N_1 \otimes_R N_2$ by setting

$$(f_1 \# f_2)(m_1, m_2) = f_1(m_1) \otimes f_2(m_2) \tag{3.4.3}$$

for $(m_1, m_2) \in M_1 \times M_2$. By the universal property, there is a unique linear map

$$f_1 \otimes f_2 : M_1 \otimes_R M_2 \to N_1 \otimes_R N_2$$

such that the following diagram commutes



The bifunctor $\otimes_R$ then assigns to each pair of morphisms f_1 and f_2, the tensor product morphism $f_1 \otimes f_2$. This completes the discussion of the bifunctor $\otimes_R$.

Exercise 3.4.3. Show that the direct sum $M \times N$ is indeed an R-module with the action defined in (3.4.1).

Exercise 3.4.4. Prove Lemma 3.4.1.

Exercise 3.4.5. Show that the free R-module $\mathcal{F}_R(X)$ generated by a set X is indeed an R-module.

Exercise 3.4.6. Show that the map defined in (3.4.3) is a bilinear map of R-modules.

Exercise 3.4.7. Show that $\otimes_R$ indeed defines a bifunctor

$$\otimes_R : \mathbf{Mod}_R \times \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_R.$$

3.5 Associative algebras

The category $\mathbf{Vect}_{\mathbb{F}}$ of $\mathbb{F}$-vector spaces has as its objects $\mathbb{F}$-modules over a field $\mathbb{F}$ and as its morphisms module homomorphisms. The tensor product construction endows this category with a bifunctor

$$\otimes_{\mathbb{F}} : \mathbf{Vect}_{\mathbb{F}} \times \mathbf{Vect}_{\mathbb{F}} \longrightarrow \mathbf{Vect}_{\mathbb{F}}$$

In the special case where our field is $\mathbb{C}$, we omit some notation and simply denote the category of $\mathbb{C}$-vector spaces by **Vect** and the corresponding bifunctor by $\otimes$. It is well-known that the bifunctor $\otimes$ endows the category **Vect** with the structure of a symmetric monoidal category, and we leave this as an exercise.

We use the term *associative algebra* to refer to a ring A with identity 1 that is also a vector space over $\mathbb{C}$ such that the action of $\mathbb{C}$ on A is bilinear in the sense that

$$r \cdot (xy) = (r \cdot x)y = x(r \cdot y)$$

for all $r \in \mathbb{C}$ and $x, y \in A$. For associative algebras A_1 and A_2, a map $\phi : A_1 \to A_2$ is called an *associative algebra homomorphism* if

(i) ϕ is a homomorphism of rings,

(ii) ϕ is a homomorphism of $\mathbb{C}$-modules, and

(iii) $\phi(1_{A_1}) = 1_{A_2}$.

The category **Alg** has as its objects associative algebras and as its morphisms homomorphisms between associative algebras.

The tensor product construction restricts to a construction on associative algebras. In particular, if A_1 and A_2 are associative algebras, then we may form the tensor product vector space $A_1 \otimes A_2$. We already showed that this space comes equipped with a $\mathbb{C}$-module structure. As a vector space, it also comes equipped with an additive structure, and a multiplicative structure is given by

$$(a_1 \otimes a_2)(b_1 \otimes b_2) := (a_1 b_1) \otimes (a_2 b_2). \tag{3.5.1}$$

It is left as an exercise to show that with these definitions the vector space $A_1 \otimes A_2$ acquires the structure of an associative algebra with unit $1 = 1_{A_1} \otimes 1_{A_2}$. Moreover, it can easily be seen that the tensor product morphism $f_1 \otimes f_2$ is a homomorphism of associative algebras for any homomorphisms f_1 and f_2. In this way, we obtain a bifunctor

$$\otimes : \mathbf{Alg} \times \mathbf{Alg} \longrightarrow \mathbf{Alg}.$$

It is not too difficult to see that the category **Alg** together with the bifunctor $\otimes$ enjoys the structure of a symmetric monoidal category. (This is, in some ways, a consequence of the fact that the more general category **Vect** is symmetric monoidal.)

A *module* for an associative algebra A is a pair (M, ϕ) where M is a vector space and

$$\begin{aligned} \phi : A &\longrightarrow \operatorname{End} M \\ a &\mapsto \phi_a \end{aligned}$$

is an associative algebra homomorphism. If (M_1, ϕ_1) and (M_2, ϕ_2) are A-modules, an *A-module homomorphism* from M_1 into M_2 is a linear map $f : M_1 \to M_2$ which is compatible with the action of A:

$$f((\phi_1)_a(m)) = (\phi_2)_a(f(m)) \quad \text{for each } a \in A \text{ and each } m \in M_1.$$

The category of A-modules, denoted $\mathbf{Mod}(A)$, has as its objects A-modules and as its morphisms A-module homomorphisms.

Lemma 3.5.1. *Let $f : A_1 \to A_2$ be a homomorphism of associative algebras and let (M, ϕ) be a module for A_2. Then $(M, \phi \circ f)$ is a module for A_1.*

Proof. This is because the composition of algebra homomorphisms is a homomorphism. □

Corollary 3.5.2. *If the algebras A_1 and A_2 are isomorphic, then the categories $\mathbf{Mod}(A_1)$ and $\mathbf{Mod}(A_2)$ are isomorphic.*

Proof. Let $\rho : A_1 \to A_2$ be an isomorphism of associative algebras. We define a functor $\rho^* : \mathbf{Mod}(A_2) \to \mathbf{Mod}(A_1)$ in the following manner. To each module (M, ϕ) for A_2, the functor ρ^* assigns the module $(M, \phi \circ \rho)$ for A_1. To each module homomorphism $g \in \operatorname{Hom}_{\mathbf{Mod}(A_2)}(M, N)$, the functor ρ^* assigns the module homomorphism $g \circ \rho \in \operatorname{Hom}_{\mathbf{Mod}(A_1)}(\rho^*(M), \rho^*(N))$. One can check that the above assignment indeed defines a functor ρ^*. Moreover, the inverse of ρ^* is $(\rho^{-1})^*$. Thus we obtain an isomorphism $\rho^* : \mathbf{Mod}(A_2) \to \mathbf{Mod}(A_1)$. □

Exercise 3.5.3. Show that the category **Vect** together with the bifunctor $\otimes$ enjoys the structure of a symmetric monoidal category. More precisely, find a distinguished vector space E, natural isomorphisms α, ρ, λ and a braiding β, such that the seven tuple $(\mathbf{Vect}, \otimes, E, \alpha, \lambda, \rho, \beta)$ is a braided monoidal category and β satisfies the property in (SMC2).

Exercise 3.5.4. Show that with the multiplication as defined in (3.5.1) the space $A_1 \otimes A_2$ acquires the structure of an associative algebra.

Exercise 3.5.5. If $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ are homomorphisms of associative algebras, show that the tensor product vector space homomorphism $f_1 \otimes f_2 : A_1 \otimes A_2 \to B_1 \otimes B_2$ is actually a homomorphism of associative algebras.

Exercise 3.5.6. Show that the category **Alg** together with the bifunctor $\otimes$ enjoys the structure of a symmetric monoidal category. More precisely, find a distinguished associative algebra E, natural isomorphisms α, ρ, λ and a braiding β, such that the seven tuple $(\mathbf{Alg}, \otimes, E, \alpha, \lambda, \rho, \beta)$ is a braided monoidal category and β satisfies the property in (SMC2).

3.6 Tensor product vertex operator algebras

The goal of this section is to show that the tensor product construction for vertex operator algebras endows the category of vertex operator algebras with a symmetric monoidal structure.

Suppose that $(V_1, Y_1, \mathbb{1}^1, d_1)$ and $(V_2, Y_2, \mathbb{1}^2, d_2)$ are vertex algebras. The *tensor product vertex algebra* of V_1 and V_2 is the quadruple $(V_1 \otimes V_2, Y, \mathbb{1}, d)$ where

(TPVA1) $V_1 \otimes V_2$ is the tensor product of the vector spaces V_1 and V_2;

(TPVA2) The map

$$Y(\cdot, x) : V_1 \otimes V_2 \longrightarrow \mathrm{End}(V_1 \otimes V_2)[[x, x^{-1}]]$$

is the unique linear map satisfying

$$Y(v^1 \otimes v^2, x) = Y_1(v^1, x) \otimes Y^2(v_2, x)$$

for $v^1 \in V_1$ and $v^2 \in V_2$;

(TPVA3) $\mathbb{1} = \mathbb{1}^1 \otimes \mathbb{1}^2$;

(TPVA4) $d = d_1 \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes d_2$.

It can be shown (see e.g. [LL]) that the quadruple $(V_1 \otimes V_2, Y, \mathbb{1}, d)$ indeed defines a vertex algebra.

If $(V_1, Y_1, \mathbb{1}^1, \omega^1)$ and $(V_2, Y_2, \mathbb{1}^2, \omega^2)$ are vertex operator algebras, then the above construction gives a tensor product vertex algebra $(V_1 \otimes V_2, Y, \mathbb{1}, d)$ where $d = \omega^1_0 \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes \omega^2_0$. Moreover, this vertex algebra carries the structure of a vertex operator algebra with Virasoro vector given by

$$\omega = \omega^1 \otimes \mathbb{1}^2 + \mathbb{1}^1 \otimes \omega^2.$$

Furthermore, if $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$ are morphisms of vertex operator algebras, then the vector space morphism $f_1 \otimes f_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ is a vertex operator algebra morphism.

In this way the tensor product construction endows the category **VOA** of vertex operator algebras with a bifunctor

$$\otimes : \mathbf{VOA} \times \mathbf{VOA} \longrightarrow \mathbf{VOA}.$$

Note that in particular, if V_1 is a vertex operator algebra of central charge c_1 and V_2 is a vertex operator algebra of central charge c_2, then the tensor product $V_1 \otimes V_2$ is a vertex operator algebra of central charge $c_1 + c_2$. This implies that $\otimes$ restricts to a bifunctor

$$\otimes : \mathbf{VOA}(c_1) \times \mathbf{VOA}(c_2) \longrightarrow \mathbf{VOA}(c_1 + c_2).$$

However, for the purposes of these notes, we often consider $\otimes$ as a bifunctor for the more general category **VOA** of vertex operator algebras.

Lemma 3.6.1. *Let $(V, Y, \mathbb{1}, \omega)$ be a vertex operator algebra and let $E = (\mathbb{C}, Y_E, 1, 0)$ be the vertex operator algebra of (2.3.2). Then the linear maps defined by*

$$\begin{aligned} \lambda_V : \mathbb{C} \otimes V &\longrightarrow V \\ k \otimes v &\mapsto kv \end{aligned}$$

and

$$\begin{aligned} \rho_V : V \otimes \mathbb{C} &\longrightarrow V \\ v \otimes k &\mapsto kv \end{aligned}$$

are isomorphisms of vertex operator algebras.

Proof. It is clear that conditions (Hom2) and (Hom3) are satisfied for both λ_V and ρ_V. We prove condition (Hom1) for simple tensors $u = k \otimes v$ for the map λ_V. The case for ρ_V is similar.

By the vacuum property (VA1), we have

$$(k \otimes v)_n = k \sum_{m \in \mathbb{Z}} \mathbb{1}_m \otimes v_{n-m-1} = k \otimes v_n.$$

If $v^1 \in \mathbb{C}$ and $v^2 \in V$, then

$$\begin{aligned} \lambda_V((k \otimes v)_n (v^1 \otimes v^2)) &= \lambda_V((k \otimes v_n)(v^1 \otimes v^2)) \\ &= \lambda_V(kv^1 \otimes v_n(v^2)) \\ &= (kv)_n (v^1 v^2) \\ &= (\lambda_V(k \otimes v))_n (\lambda_V(v^1 \otimes v^2)), \end{aligned}$$

as desired.

Finally, it is clear that the maps λ_V and ρ_V are bijective maps, and hence, they are isomorphisms of vertex operator algebras. □

Corollary 3.6.2. *The functors $\mathcal{L}_E^{\otimes}$ and $\mathcal{R}_E^{\otimes}$ are naturally isomorphic to the identity functor $\mathcal{I}$:* **VOA** $\to$ **VOA**.

Proof. The natural isomorphisms are given by λ and ρ respectively. □

Let V_1, V_2, V_3 be vertex operator algebras. By the universal mapping property, the bilinear map

$$A_{V_1,V_2,V_3} : (V_1 \otimes V_2) \times V_3 \longrightarrow V_1 \otimes (V_2 \otimes V_3)$$
$$(v^1 \otimes v^2, v^3) \mapsto v^1 \otimes (v^2 \otimes v^3)$$

induces a well-defined linear map

$$\alpha_{V_1,V_2,V_3} : (V_1 \otimes V_2) \otimes V_3 \longrightarrow V_1 \otimes (V_2 \otimes V_3)$$
$$(v^1 \otimes v^2) \otimes v^3 \mapsto v^1 \otimes (v^2 \otimes v^3).$$

It is routine to check that α_{V_1,V_2,V_3} is an isomorphism of vector spaces (e.g. by constructing an inverse). And moreover, we have the following stronger result, which we leave as a simple exercise.

Lemma 3.6.3. *The map α_{V_1,V_2,V_3} is an isomorphism of vertex operator algebras.*

Corollary 3.6.4. *The functors $\mathcal{A}_L^{\otimes}$ and $\mathcal{A}_R^{\otimes}$ are naturally isomorphic.*

Proof. The natural isomorphism is given by α. □

Corollary 3.6.5. *The six-tuple $(\mathbf{VOA}, \otimes, E, \alpha, \lambda, \rho)$ is a monoidal category.*

Proof. We have shown axioms (MC1) through (MC3). It is a well-known fact that the pentagon diagram (MC4) and triangle diagram (MC5) commute for the category of $\mathbb{Z}$-graded vector spaces equipped with the graded tensor product, and hence we obtain the result. □

Let $(V_1, Y_1, \mathbb{1}^1, \omega^1)$ and $(V_2, Y_2, \mathbb{1}^2, \omega^2)$ be vertex operator algebras. By the universal mapping property, the bilinear map

$$B_{V_1,V_2} : V_1 \oplus V_2 \longrightarrow V_2 \otimes V_1$$
$$(v^1, v^2) \mapsto v^2 \otimes v^1$$

induces a well-defined linear map

$$\beta_{V_1,V_2} : V_1 \otimes V_2 \longrightarrow V_2 \otimes V_1$$
$$v^1 \otimes v^2 \mapsto v^2 \otimes v^1.$$

Lemma 3.6.6. *The map $\beta_{V_1,V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1$ is an isomorphism of vertex operator algebras.*

Proof. It is clear that conditions (Hom2) and (Hom3) are satisfied. We prove condition (Hom1) for simple tensors $u = u^1 \otimes u^2 \in V_1 \otimes V_2$.

Let $u^1 \in V_1$ and $u^2 \in V_2$. Observe that

$$(u^1 \otimes u^2)_k = \sum_{m\in\mathbb{Z}} u^1_m \otimes u^2_{k-m-1}.$$

If $v^1 \in V_1$ and $v^2 \in V_2$, then

$$\begin{aligned}\beta_{V_1,V_2}\left((u^1 \otimes u^2)_k(v^1 \otimes v^2)\right) &= \beta_{V_1,V_2}\left(\sum_{m\in\mathbb{Z}}(u^1_m \otimes u^2_{k-m-1})(v^1 \otimes v^2)\right)\\ &= \sum_{m\in\mathbb{Z}} \beta_{V_1,V_2}(u^1_m(v^1) \otimes u^2_{k-m-1}(v^2))\\ &= \sum_{m\in\mathbb{Z}} u^2_{k-m-1}(v^2) \otimes u^1_m(v^1)\\ &= (u^2 \otimes u^1)_k(v^2 \otimes v^1)\\ &= (\beta_{V_1,V_2}(u^2 \otimes u^1))_k(\beta_{V_1,V_2}(v^1 \otimes v^2)),\end{aligned}$$

as desired.

The inverse of β_{V_1,V_2} is β_{V_2,V_1}. □

Corollary 3.6.7. *The bifunctors $\otimes$ and $\otimes^{op}$ are naturally isomorphic.*

Proof. The isomorphism is given by β. □

Corollary 3.6.8. *The seven-tuple $(\mathbf{VOA}, \otimes, E, \alpha, \lambda, \rho, \beta)$ is a symmetric monoidal category.*

Proof. The axioms (BCM1) are stated in Corollary 3.6.5 and the axiom (BCM2) is stated in Corollary 3.6.7. It is well-known that the first (BCM3) and second (BCM4) hexagon diagrams commute for the category of $\mathbb{Z}$-graded vector spaces equipped with the graded tensor product, and thus we see that the seven-tuple $(\mathbf{VOA}, \otimes, E, \alpha, \lambda, \rho, \beta)$ is a braided monoidal category. The fact that the inverse of β_{V_1,V_2} is β_{V_2,V_1} shows that, in fact, the seven-tuple $(\mathbf{VOA}, \otimes, E, \alpha, \lambda, \rho, \beta)$ is a symmetric monoidal category. □

Exercise 3.6.9. Show that if V_1 is a vertex operator algebra of central charge c_1 and V_2 is a vertex operator algebra of central charge c_2, then the central charge c of the tensor product $V_1 \otimes V_2$ is $c = c_1 + c_2$.

Exercise 3.6.10. Show that the map

$$\begin{aligned}\alpha_{V_1,V_2,V_3} : (V_1 \otimes V_2) \otimes V_3 &\longrightarrow V_1 \otimes (V_2 \otimes V_3)\\ (v_1 \otimes v_2) \otimes v_3 &\mapsto v_1 \otimes (v_2 \otimes v_3)\end{aligned}$$

is an isomorphism of vertex operator algebras.

Chapter 4

Zhu's algebra

Following [Z], we construct a functor Λ from the category **VOA** of vertex operator algebras to the category **Alg** of associative $\mathbb{C}$-algebras.

4.1 Construction of the functor

To each vertex operator algebra V, the map Λ assigns an associative algebra $\Lambda(V)$ formed in the following way. For any $a, b \in V$ with a homogeneous, let $a \circ b$ denote the element

$$a \circ b = \mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(a)}}{z^2} Y(a,z)b \right) \in V.$$

Let $O(V)$ denote the subspace of V spanned by all elements of the form $a \circ b$ for some $a, b \in V$ with a homogeneous. Let $\Lambda(V)$ denote the quotient space $V/O(V)$. For $a \in V$, we let $[a]$ denote the equivalence class of a in $\Lambda(V)$.

For $a, b \in V$ with a homogeneous, let $a * b$ denote the element

$$a * b = \mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(a)}}{z} Y(a,z)b \right) \in V.$$

The map $*$ defines a bilinear map from $V \times V$ into V. One can show that $O(V)$ is an ideal under the operation $*$ (cf. [Z]), and hence $\Lambda(V)$ acquires a well-defined induced bilinear map:

$$[a] * [b] := [a * b],$$

which we still denote by $*$. It is shown in [Z], that $[\mathbb{1}]$ becomes a unit for $\Lambda(V)$ under this multiplicative structure. It is also shown that the multiplication is associative.

To a homomorphism $f : V \to W$ of vertex operator algebras V and W, the map Λ assigns a map of associative algebras $f_\Lambda : \Lambda(V) \to \Lambda(W)$ defined in the following manner:

$$f_\Lambda([v]) = [f(v)] \quad \text{for each } v \in V.$$

Proposition 4.1.1. *For any homomorphism $f : V \to W$ of vertex operator algebras V and W, the induced map $f_\Lambda : \Lambda(V) \to \Lambda(W)$ is a well-defined homomorphism of associative algebras.*

Proof. We first show that the map f_Λ is well-defined. Let $v, v' \in V$, and suppose that $v' = v + w$ for some $w \in O(V)$. Then w is of the form

$$w = \mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(a)}}{z^2} Y(a,z)b \right)$$

for some $a, b \in V$. Since f is a homomorphism of vertex operator algebras, it follows that

$$\begin{aligned} f(v+w) &= f(v) + f(w) \\ &= f(v) + f\left(\mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(a)}}{z^2} Y(a,z)b \right) \right) \\ &= f(v) + \mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(f(a))}}{z^2} Y(f(a),z)f(b) \right) \\ &= f(v) + f(a) \circ f(b). \end{aligned}$$

This shows that $f(v') \in [f(v)]$ as desired.

To see that f_Λ is a homomorphism of associative algebras, first note that

$$f_\Lambda([\mathbb{1}_V]) = [f(\mathbb{1}_V)] = [\mathbb{1}_W].$$

Moreover, for any $a, b \in V$, we have

$$\begin{aligned} f_\Lambda([a] * [b]) &= f_\Lambda([a * b]) \\ &= [f(a * b)] \\ &= \left[f\left(\mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(a)}}{z} Y(a,z)b \right) \right) \right] \\ &= \left[\mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(f(a))}}{z} Y(f(a),z)f(b) \right) \right] \\ &= [f(a) * f(b)] \\ &= [f(a)] * [f(b)] \end{aligned}$$

$$= f_\Lambda([a]) * f_\Lambda([b]),$$

as desired. □

Corollary 4.1.2. *The map* $\Lambda : \mathbf{VOA} \to \mathbf{Alg}$ *is a functor.* □

4.2 Symmetric monoidal properties

A natural question is to ask how the functor Λ behaves under the tensor product structure. We will show, perhaps unsurprisingly, that Λ is compatible with the tensor product structure.

Lemma 4.2.1. *Let V and W be vertex operator algebras. If a belongs to $O(V)$ and b is any vector in W, then the element $a \otimes b$ belongs to $O(V \otimes W)$.*

Proof. Let $v, v' \in V$ and $w, w' \in W$. Recall that

$$\begin{aligned} v \circ v' &= \mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(v)}}{z^2} Y(v,z)v' \right) \\ &= \sum_{n \in \mathbb{N}} \binom{\mathrm{wt}(v)}{n} v_{n-2}(v'). \end{aligned}$$

Observe that

$$\begin{aligned} &v \otimes w \circ v' \otimes w' \\ &= \mathrm{Res}_z \left(\sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \binom{\mathrm{wt}(v \otimes w)}{n} v_m(v') \otimes w_k(w') z^{n-m-k-4} \right). \end{aligned} \tag{4.2.1}$$

Taking $w = \mathbb{1}$ in (4.2.1), we obtain

$$\begin{aligned} &(v \otimes \mathbb{1}) \circ (v' \otimes w) \\ &= \sum_{n \in \mathbb{N}} \binom{\mathrm{wt}(v)}{n} v_{n-2}(v') \otimes w' \\ &= (v \circ v') \otimes w. \end{aligned}$$

Since v and v' are arbitrary, this proves the result. □

Theorem 4.2.2. *Let V and W be vertex operator algebras. Define a map*

$$\begin{aligned} \Phi_{V,W} : \Lambda(V) \times \Lambda(W) &\to \Lambda(V \otimes W) \\ ([v],[w]) &\mapsto [v \otimes w]. \end{aligned}$$

Then $\Phi_{V,W}$ is a well-defined bilinear map of vector spaces.

Proof. We first show that $\Phi_{V,W}$ is well-defined. Let $v \in V$ and $w \in W$ and suppose that $v' = v + a$ for some $a \in O(V)$ and $w' = w + b$ for some $b \in O(W)$. Then observe that

$$\begin{aligned} v' \otimes w' &= (v + a) \otimes (w + b) \\ &= v \otimes w + v \otimes b + a \otimes w + a \otimes b. \end{aligned}$$

By the Lemma 4.2.1, the terms $v \otimes b$, $a \otimes w$ and $a \otimes b$ all belong to $O(V \otimes W)$. Therefore, $v' \otimes w'$ belongs to $[v \otimes w]$ as desired.

To see that the map is bilinear, observe that if $c \in \mathbb{C}$, then

$$\begin{aligned} \Phi_{V,W}(c[v] + [v'], [w]) &= \Phi_{V,W}([cv + v'], [w]) \\ &= [(cv + v') \otimes w] \\ &= [c \cdot (v \otimes w) + v' \otimes w] \\ &= c \cdot [v \otimes w] + [v' \otimes w] \\ &= c \cdot \Phi_{V,W}([v], [w]) + \Phi_{V,W}([v'], [w]). \end{aligned}$$

Showing linearity in the second slot is similar. □

By the universal mapping property, we obtain a well-defined linear map of vector spaces described by

$$\begin{aligned} \phi_{V,W} : \Lambda(V) \otimes \Lambda(W) &\to \Lambda(V \otimes W) \\ [v] \otimes [w] &\mapsto [v \otimes w]. \end{aligned}$$

We now show that $\phi_{V,W}$ is a homomorphism of associative algebras.

Proposition 4.2.3. *The map $\phi_{V,W} : \Lambda(V) \otimes \Lambda(W) \to \Lambda(V \otimes W)$ is a homomorphism of associative algebras.*

Proof. For any $v, v' \in V$ and $w, w' \in W$, we have

$$\begin{aligned} \phi_{V,W}(([v] \otimes [w]) * ([v'] \otimes [w'])) &= \phi_{V,W}(([v] * [v']) \otimes ([w] * [w'])) \\ &= \phi_{V,W}([v * v'] \otimes [w * w']) \\ &= [(v * v') \otimes (w * w')] \\ &= [(v \otimes w) * (v' \otimes w')] \\ &= [v \otimes w] * [v' \otimes w'] \\ &= \phi_{V,W}([v] \otimes [w]) * \phi_{V,W}([v'] \otimes [w']), \end{aligned}$$

as desired. □

A proof of the following lemma can be found in [Z].

Lemma 4.2.4. *Let v, v' be elements of a vertex operator algebra V. For $n \in \mathbb{Z}$, let $v \circ (z^n v')$ denote the element*

$$v \circ (z^n v') = \mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(v)}}{z^{2-n}} Y(v,z)v' \right).$$

Then for each $k \in \mathbb{N}$, the element $v \circ (z^{-k} v')$ belongs to $O(V)$. □

Lemma 4.2.5. *Let V be a vertex operator algebra and let $v, v' \in V$. Let*

$$v(m,k)(v') = \sum_{n \in \mathbb{N}} \binom{\mathrm{wt}(v)}{n} v_{n+m-k-3}(v').$$

Then for each $m \in \mathbb{N}$ and $k \geq m-1$, the element $v(m,k)(v')$ belongs to $O(V)$.

Proof. Fix $m \in \mathbb{N}$ and $k \geq m-1$. By Lemma 4.2.4, the sum

$$\sum_{n \in \mathbb{N}} \binom{\mathrm{wt}(v)}{n} v_{m+n-k-3}(v') = v \circ (z^{m-k+1} v')$$

belongs to $O(V)$. □

Lemma 4.2.6. *For vertex operator algebras V and W, the equality*

$$O(V \otimes W) = O(V) \otimes W + V \otimes O(W)$$

holds.

Proof. Recall that $O(V) \otimes W \subset O(V \otimes W)$ and $V \otimes O(W) \subset O(V \otimes W)$ by Lemma 4.2.1. It follows that the sum of subspaces $O(V) \otimes W + V \otimes O(W)$ is a subset of $O(V \otimes W)$.

Suppose that $\mathbf{a} \in O(V \otimes W)$. Then there are elements $v, v' \in V$ and $w, w' \in W$ such that

$$\begin{aligned} \mathbf{a} &= (v \otimes w) \circ (v' \otimes w') \\ &= \mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(v \otimes w)}}{z^2} Y(v \otimes w, z)(v' \otimes w') \right) \\ &= \mathrm{Res}_z \left(z^{-2} (1+z)^{\mathrm{wt}(v)} Y(v,z)v' \otimes (1+z)^{\mathrm{wt}(w)} Y(w,z)w' \right) \\ &= \mathrm{Res}_z \left(\sum_{m,n \in \mathbb{N}} \sum_{k,l \in \mathbb{Z}} \binom{\mathrm{wt}(v)}{n} \binom{\mathrm{wt}(w)}{m} v_l(v') \otimes w_k(w') z^{n+m-k-l-4} \right). \end{aligned}$$

Let

$$\mathfrak{v} = \sum_{m \in \mathbb{N}} \binom{\mathrm{wt}(w)}{m} \left(\sum_{k \geq m-1} v(m,k)(v') \otimes w_k(w') \right)$$

$$\mathfrak{w} = \sum_{n\in\mathbb{N}} \binom{\mathrm{wt}(v)}{n} \left(\sum_{l\geq n-1} v_l(v') \otimes w(n,l)(w') \right).$$

It is not too difficult to verify that

$$\mathbf{a} = \mathfrak{v} + \mathfrak{w}.$$

Moreover, by Lemma 4.2.5, the element $\mathfrak{v}$ belongs to $O(V) \otimes W$ and the element $\mathfrak{w}$ belongs to $V \otimes O(W)$. ☐

Theorem 4.2.7. *The map $\phi_{V,W} : \Lambda(V) \otimes \Lambda(W) \to \Lambda(V \otimes W)$ is an isomorphism of associative algebras.*

Proof. It is clear that $\phi_{V,W}$ is surjective, so it remains only to show that $\phi_{V,W}$ is injective.

Suppose that $\mathbf{a} \in V \otimes W$ belongs to $O(V \otimes W)$. Then Lemma 4.2.6 shows that $\mathbf{a}$ is of the form

$$\mathbf{a} = \sum_i v'^{(i)} \otimes w^{(i)} + \sum_j v^{(j)} \otimes w'^{(j)}$$

for some $v'^{(i)} \in O(V)$, $w^{(i)} \in W$, $v^{(j)} \in V$, and $w'^{(j)} \in O(W)$. It follows that

$$\sum_i [v'^{(i)}] \otimes [w^{(i)}] + \sum_j [v^{(j)}] \otimes [w'^{(j)}]$$

is a representative of the zero element of $\Lambda(V) \otimes \Lambda(W)$. This shows that $\phi_{V,W}$ is injective, as desired. ☐

Furthermore, we claim that the map $\phi_{V,W}$ is natural, in the following sense.

Proposition 4.2.8. *Let $f : V \to V'$ and $g : W \to W'$ be homomorphisms of vertex operator algebras. Then the following diagram commutes*

$$\begin{array}{ccc} \Lambda(V) \otimes \Lambda(W) & \xrightarrow{\phi_{V,W}} & \Lambda(V \otimes W) \\ \downarrow{\scriptstyle f_\Lambda \otimes g_\Lambda} & & \downarrow{\scriptstyle (f\otimes g)_\Lambda} \\ \Lambda(V') \otimes \Lambda(W') & \xrightarrow{\phi_{V',W'}} & \Lambda(V' \otimes W'). \end{array}$$

Proof. Let $v \in V$ and $w \in W$. Then observe that

$$\begin{aligned} \phi_{V',W'}((f_\Lambda \otimes g_\Lambda)([v] \otimes [w])) &= \phi_{V',W'}(f_\Lambda([v]) \otimes g_\Lambda([w])) \\ &= \phi_{V',W'}([f(v)] \otimes [g(w)]) \\ &= [f(v) \otimes g(w)] \\ &= [(f \otimes g)(v \otimes w)] \end{aligned}$$

$$= (f \otimes g)_\Lambda([v \otimes w])$$
$$= (f \otimes g)_\Lambda(\phi_{V,W}([v] \otimes [w])),$$

as desired. ☐

The proof of the following lemma is left as an exercise.

Lemma 4.2.9. *Let $\epsilon : \mathbb{C} \to \Lambda(E)$ denote the map of associative algebras described by*

$$\epsilon : \mathbb{C} \to \Lambda(E)$$
$$k \mapsto k[\mathbb{1}]$$

where E is the vertex operator algebra of (2.3.2)*. Then ϵ is a homomorphism of associative algebras.* ☐

Theorem 4.2.10. *The functor Λ is a lax monoidal functor from the monoidal category $(\mathbf{VOA}, \otimes, E, \alpha, \lambda, \rho)$ into $(\mathbf{Alg}, \otimes, \mathbb{C}, \alpha, \lambda, \rho)$.*

Proof. Let V, W, Z be vertex operator algebras and let $v \in V, w \in W, z \in Z$. Observe that

$$\begin{aligned}
&\phi_{V\otimes W,Z}(\phi_{X,Y} \otimes \mathrm{id}_{\Lambda(Z)}([v] \otimes [w] \otimes [z])) \\
&= \phi_{V\otimes W,Z}([v \otimes w] \otimes [z]) \\
&= [v \otimes w \otimes z] \\
&= \phi_{V,W\otimes Z}([v] \otimes [w \otimes z]) \\
&= (\phi_{V,W\otimes Z}(\mathrm{id}_{\Lambda(V)} \otimes \phi_{W,Z}([v] \otimes [w] \otimes [z])),
\end{aligned}$$

showing that (LM1) holds.

Also note that

$$\begin{aligned}
&\Lambda(\lambda_V)(\phi_{E,V}(\epsilon \otimes \mathrm{id}_{\Lambda(V)}(k \otimes [v]))) \\
&= \Lambda(\lambda_V)(\phi_{E,V}(k[\mathbb{1}] \otimes [v])) \\
&= \Lambda(\lambda_V)(\phi_{E,V}([k] \otimes [v])) \\
&= \Lambda(\lambda_V)([k \otimes v]) \\
&= [kv] \\
&= k[v] \\
&= \lambda_{\Lambda(V)}(k \otimes [v]),
\end{aligned}$$

showing that (LM2) holds. The proof of (LM3) is similar. ☐

Corollary 4.2.11. *The functor Λ is symmetric monoidal.*

Proof. It is routine to verify that the diagram (BMF) commutes. ☐

Exercise 4.2.12. Prove Lemma 4.2.9.

4.3 Related module functors

For each vertex operator algebra V, the functor Λ induces a functor Ω_V on the level of modules, which has played an important role in the classification of vertex operator algebras and their modules. The goal of this section is to construct this functor and outline some of its properties. More specifically, for a vertex operator algebra $(V, Y, \mathbb{1}, \omega)$, we follow [Z] to construct a functor $\Omega_V : \mathbf{Weak}(V) \to \mathbf{Mod}(\Lambda(V))$ from the category of weak modules for V to the category of modules for the associative algebra $\Lambda(V)$.

Let (M, Y_M) be a weak module for a vertex operator algebra $(V, Y, \mathbb{1}, \omega)$. Recall that this means that $(M, Y_M, L(-1))$ is a module for the vertex algebra $(V, Y, \mathbb{1}, L(-1))$. We then let $\Omega_V(M)$ denote the space of lowest weight vectors in M:

$$\Omega_V(M) = \{v \in M : a_k v = 0 \text{ for homogeneous } a \text{ and each } k \geq \mathrm{wt}(a)\}.$$

The space $\Omega_V(M)$ acquires the structure of a module for the algebra $\Lambda(V)$ in the following manner.

Define a linear map $\mu' : V \to \mathrm{End}\, M$ by setting $\mu'(a) = a_{\mathrm{wt}(a)-1}$ for homogeneous $a \in V$. In [Z], it is shown that $O(V)$ is a subset of the kernel of μ', and hence μ' descends to a well-defined linear map $\mu : \Lambda(V) \to \mathrm{End}\, M$ on the quotient space, which is described by $\mu([a]) = a_{\mathrm{wt}(a)-1}$. In [Z, L94], we find the following result.

Theorem 4.3.1. *Let (M, Y_M) be a weak module for a vertex operator algebra $(V, Y, \mathbb{1}, \omega)$. Then the pair $(\Omega_V(M), \mu)$ is a module for the associative algebra $\Lambda(V)$.*

Remark 4.3.2. In his original work [Z], Zhu defines the space $\Omega_V(M)$ in a slightly different manner. Indeed, he shows that for a *weak-admissible* module $M = \bigoplus_{n \in \mathbb{N}} M_n$, the homogeneous subspace M_0 is a module for the associative algebra $\Lambda(V)$. It was realized (e.g. by Li in his thesis [L94]) that more generally, for any *weak* module M, the space $\Omega_V(M)$ acquires the structure of a $\Lambda(V)$-module. In particular, if M is a weak-admissible module, then the *top homogeneous weight space* M_0 will be a subspace of the space $\Omega_V(M)$, and thus M_0 is a submodule of the $\Lambda(V)$-module $\Omega_V(M)$. Thus, Zhu's original formulation is extended by Li's Ω_V construction.

Moreover, we can let Ω_V act on morphisms in the following manner.

Proposition 4.3.3. *Let V be a vertex operator algebra and let M and N be two weak modules for V. Let $f : M \to N$ be a weak V-module homomorphism. Then the image of the restriction $f_{\Omega_V} = f|_{\Omega_V(M)}$ is contained within $\Omega_V(N)$ and f is a well-defined $\Lambda(V)$-module homomorphism.*

Proof. We first show that the image of f_{Ω_V} is a subset of $\Omega_V(N)$. Let v be an element of $\Omega_V(M)$. Recall that this means that $a_k v = 0$ for each homogeneous $a \in V$ and $k \geq \mathrm{wt}(a)$. Using (2.3.1), we see that

$$a_k(f(v)) = f(a_k(v)) = f(0) = 0$$

for each homogeneous $a \in V$ and $k \geq \mathrm{wt}(a)$. It follows that $f(v)$ belongs to $\Omega_V(N)$, as desired.

We now show that f_{Ω_V} is an $\Lambda(V)$-module homomorphism. Let $(\Omega_V(M), \mu_M)$ and $(\Omega_V(N), \mu_N)$ denote the respective modules for $\Lambda(V)$. For $v \in \Omega_V(M)$ and homogeneous $a \in V$, we compute

$$\begin{aligned} \mu_N([a])f(v) &= a_{\mathrm{wt}(a)-1}(f(v)) \\ &= f(a_{\mathrm{wt}(a)-1}(v)) \\ &= f(\mu_M([a])v), \end{aligned}$$

showing that f is a $\Lambda(V)$-module homomorphism. □

Corollary 4.3.4. *The assignment Ω_V defines a functor from the category **Weak**(V) into the category **Mod**$(\Lambda(V))$.*

Proof. Exercise. □

Moreover, one can construct a functor $\Gamma_V : \mathbf{Mod}(\Lambda(V)) \to \mathbf{Adm}(V) \subset \mathbf{Weak}(V)$ which acts as an inverse for the functor Ω_V, when restricted to the category of weak-admissible V-modules. In fact, there are at least two different formulations of this functor, namely Zhu's original formulation in [Z] and Li's approach in [L94]. Zhu's formulation uses recurrent formulas for correlation functions on the sphere to define a representation of V whose action is completely determined by the action of the algebra $\Lambda(V)$ on the module $\Omega_V(M)$. Li's formulation involves constructing the corresponding affine vertex algebra $\widehat{V}$ and letting a quotient of this (which carries the structure of a Lie algebra) act on a module for the algebra $\Lambda(V)_{Lie}$, thereby obtaining an induced module from which we can obtain the desired V-module as a quotient module. A complete discussion of either of these approaches, however, would take us too far from course. Instead, let us summarize the main result below.

Theorem 4.3.5. *The functors Ω_V and Γ_V induce a bijection*

$$\{\textit{simple } \Lambda(V)\textit{-modules}\}/\sim \;\longleftrightarrow\; \{\textit{simple weak-admissible } V\textit{-modules}\}/\sim$$

between the isomorphism classes of simple objects in the categories of $\Lambda(V)$-modules and weak-admissible V-modules.

This result is somewhat striking. It says that this Zhu's algebra contains essentially all of the information of the original vertex algebra V. Indeed there is a bijective correspondence between simple objects in certain module categories. Thus, this result has been particularly important in the representation theory of vertex algebras (and more specifically vertex operator algebras) because the corresponding Zhu's algebra is an associative algebra, and the representation theory of associative algebras is arguably better understood.

Exercise 4.3.6. Verify that Ω_V indeed defines a functor.

Chapter 5

Geometric motivation

A *field theory* in physics usually refers to a physical theory about how certain fields (that is, scalar fields) interact with one another and with matter. The term *classical field theory* is usually reserved for those theories that are concerned with (i) the field of gravity or (ii) fields associated with electrodynamics. These types of field theories have been well developed (both mathematically and physically).

The data of a classical field theory are a smooth manifold M, called the *space of states*, together with a smooth *action* $S : M \to \mathbb{R}$. The physics given by this field theory consists of those fields that are critical points for the action S, that is,

$$\mathrm{Crit}(S) = \{\phi \in M : dS(\phi) = 0\}.$$

The set of critical points is also described as the space of solutions to the *Euler-Lagrange equation(s)* for the action S.

For example, in the case of gravity, there is a corresponding classical field theory that declares which paths $\phi : [a, b] \to X$ a particle of mass m may take if it were to travel from a point $x \in X$ in a manifold X to a point $y \in X$ while being subject to *Newton's equation*

$$m \cdot \ddot{\phi}(t) = F(\phi(t)).$$

The space of states in this case is the set

$$M_{x,y} = \{\phi \in C^\infty([a, b], X) : \phi(a) = x, \phi(b) = y\}$$

of paths in X with starting point x and terminal point y, and the action is

$$S(\phi) = \int_a^b L(\phi(t), \dot{\phi}(t))dt,$$

a certain integration function. (Here, L denotes the *Lagrangian* function from mechanics.) The allowable paths plucked out by Newton's equation are realized as exactly those paths which are critical points for the action $S : M_{x,y} \to \mathbb{R}$.

On the other hand, the term *quantum field theory* is reserved for an obvious setting, but lacks an obvious rigorous (mathematical) definition. Indeed, a quantum field theory should be a field theory that constructs models for particle (or subatomic particle) interactions subject to certain quantum mechanical phenomena. However, how exactly one goes about constructing such a theory has been difficult to pin down.

Nevertheless, there are many approaches to quantum field theories. Among them are

- *Vertex operator algebras* [H91, H97, K],
- *Functorial field theories* [S88, S89, A, HK04, HK05], and
- *Factorization algebras* [BD].

These approaches interact and intersect not only with each other, but with various other areas of mathematics and physics. These interactions make these areas of study useful not only for their implications in theoretical physics, but also for their potential to offer better understanding of various other fields of mathematics.

Some approaches have even developed in tandem a method of *quantization*, by which one can try to quantize a classical field theory to arrive at a given quantum field theory [AE]. The hope is that a quantum field theory together with a quantization method will tie classical field theories and quantum field theories together into one unified framework.

Even better than just a quantum field theory, however, would be a *complete* quantum field theory, which would incorporate all known particles and forces. In particular, the force of major concern is the gravitational force, and it has not found its way into any accepted quantum field theory. Nevertheless, several theories, including string theories, have been postulated to incorporate this elusive force.

A *string theory* is a particular type of quantum field theory which is based upon the principle that particles are vibrating strings in space-time. As strings travel and interact in time, they are assumed to sweep out smooth manifolds, called *world sheets*. The classic example of a world sheet is that of a *pair of pants*, whereby two strings interact to form one. The world sheets are usually given a certain sewing operation, which says how world sheets should be sewn together at their tubes and which amounts to a geometric way of studying consecutive string interactions. This geometric picture is the setting for several approaches to quantum field theories, including those constructions given by Segal [S88, S89] and vertex operator algebras [H97], which are actually construction of a specific piece (namely the genus-zero piece) of Segal's construction.

Such theories that work in this setting of two-dimensional manifolds together with sewing are usually then simplified by asserting that all geometry be studied up to conformal equivalence. That is, if a world sheet may be mapped onto another by an invertible function which is angle preserving, then the two world sheets are assumed to be equivalent. One then shows that the sewing operation on world sheets induces a well-defined sewing operation on conformal equivalence classes of world sheets, and one studies the space of these equivalence classes of world sheets together with this induced sewing operation. When such assumptions are made, it is usually said that one is formulating a *conformal field theory*, or more precisely, a *two-dimensional conformal field theory*.

Vertex operator algebras arise in this setting of two-dimensional conformal field theory. However, a vertex operator algebra is not a full conformal field theory in the usual sense. Instead, a vertex operator algebra is an algebraic object realizing a basic sub-structure within the geometric picture of world sheets and sewing, namely the sub-structure of genus-zero world sheets and their sewing.

More precisely, following [H97], if our attention is restricted only to those equivalence classes of world sheets of genus zero together with a sewing operation (which produces world sheets of genus zero), then a classical uniformization theorem from complex analysis is relevant. A version of this theorem, usually dubbed the *Riemann Uniformization Theorem*, states that any compact, genus zero Riemann surface is conformally equivalent to the *standard sphere* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the one-point compactification of the complex numbers. This result gives a way of translating the conformal data of a genus-zero world sheet with n incoming tubes and 1 outgoing tube to the data of a certain standard *sphere with tubes of type* $(1, n)$

$$(\widehat{\mathbb{C}}; \infty, z_1, \ldots, z_{n-1}, 0; (B_\infty^{r_0}, \psi_0), (B_{z_1}^{r_1}, \psi_1), \ldots, B_0^{r_n}, \psi_n))$$

where the z_i are distinct nonzero complex numbers, the $B_{z_i}^{r_i}$ are open neighborhoods of these numbers, and the $\psi_i : B_{z_i}^{r_i} \to \mathbb{C}$ are local analytic injective maps vanishing at these punctures. This translation of data implies that the moduli space $\mathcal{K}(n)$ of equivalence classes of spheres with tubes of type $(1, n)$ may be identified with a certain subset of infinite sequences of complex numbers, whose coordinates give (i) the locations of the punctures and (ii) the germs of the analytic injective maps vanishing at these punctures (by taking the coefficients of their Laurent series expansions).

The sewing operation for world sheets induces a sewing operation for these moduli spaces

$$\mathcal{K}(m) \times \mathcal{K}(n) \longrightarrow \mathcal{K}(m+n-1)$$
$$(\mathfrak{S}_1, \mathfrak{S}_2) \mapsto \mathfrak{S}_1\, {}_i\infty_0\, \mathfrak{S}_2,$$

which states how the 0-th tube of $\mathfrak{S}_2$ may be sewn into the i-th tube of $\mathfrak{S}_1$. The resulting sewn sphere with tubes is realized as a solution to a certain *sewing equation*, which gives algebraic relations on the two input sequences of complex numbers. One can show that not only does a formal algebraic solution to this equation exist, but also that it is unique and moreover that it defines an analytic solution, in the sense that certain power series described by the resulting sequence of complex numbers converge in necessary neighborhoods of the complex plane. Present in the formal solution to this equation is a certain series Γ which in some sense describes the sewing structure.

A *geometric vertex operator algebra* is an algebraic-like object of vector space homomorphisms realizing the sewing structure of the above moduli spaces. The homomorphisms at play are those of the form

$$\underbrace{V \otimes \cdots \otimes V}_{n \text{ times}} \longrightarrow \overline{V}$$

from the n-fold tensor product of a vector space V to its algebraic completion $\overline{V}$. Given two homomorphisms $f \in \mathrm{Hom}(V^{\otimes m}, \overline{V})$ and $g \in \mathrm{Hom}(V^{\otimes n}, \overline{V})$, one may sometimes form a new homomorphism

$$f\ {}_i{*}_0\ g : \underbrace{V \otimes \cdots \otimes V}_{m+n-1 \text{ times}} \longrightarrow \overline{V} \tag{5.0.1}$$

obtained by "sewing" the function g into the i-th component for f. The idea is that the equivalence class $[\mathfrak{S}]$ of a sphere $\mathfrak{S}$ with tubes of type $(1, n)$ should correspond to a homomorphism $\nu_n([\mathfrak{S}]) \in \mathrm{Hom}(V^{\otimes n}, \overline{V})$ and that the sewing structure in (5.0.1) should somehow reflect the sewing of world sheet interactions. More formally, this is to say that a geometric vertex operator algebra consists of roughly the following:

- A vector space V,
- A sequence $\nu = \{\nu_n\}_{n \in \mathbb{N}}$ of maps

$$\nu_n : \mathcal{K}(n) \longrightarrow \mathrm{Hom}(V^{\otimes n}, \overline{V})$$

subject to certain conditions, among which is that the following diagram should "almost" commute.

$$\begin{array}{ccc} \mathcal{K}(m) \times \mathcal{K}(n) & \xrightarrow{\ {}_i\infty_0\ } & \mathcal{K}(m+n-1) \\ \downarrow{\scriptstyle \nu_m \times \nu_n} & & \downarrow{\scriptstyle \nu_{m+n-1}} \\ \mathrm{Hom}(V^{\otimes m}, \overline{V}) \times \mathrm{Hom}(V^{\otimes n}, \overline{V}) & \xrightarrow{\ {}_i{*}_0\ } & \mathrm{Hom}(V^{\otimes m+n-1}, \overline{V}) \end{array}$$

More precisely, the diagram should commute up to a factor of $e^{c\Gamma}$, for some complex number c which is called the *central charge* of the geometric vertex operator algebra (V, ν).

5.1 Spheres with tubes

The goal of this section is to make rigorous the geometry of string interactions using the language of complex analytic manifolds. Following Huang [H97], we introduce a notion of a sphere with tubes, which should correspond to a genus-zero world sheet of string interactions in space-time.

Let M be a connected Hausdorff topological space. An *n-dimensional complex chart* on M is a pair (U, ϕ) where U is an open subset of M and $\phi : U \to \mathbb{C}^n$ is a homeomorphism of U onto some open subset of $\mathbb{C}^n$. A *complex analytic atlas of n-dimensional complex charts* is a collection of n-dimensional complex charts $\mathcal{A} = (U_\alpha, \phi_\alpha)_{\alpha \in A}$ such that

(i) the union $\cup_{\alpha \in A} U_\alpha$ covers M and

(ii) whenever $\alpha, \beta \in A$ are such that $U_\alpha \cap U_\beta \neq \varnothing$, the map

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\beta \cap U_\alpha) \longrightarrow \phi_\beta(U_\beta \cap U_\alpha)$$

is complex analytic.

A connected Hausdorff topological space M together with a complex analytic atlas $\mathcal{A}$ of n-dimensional complex charts is called an *n-dimensional complex manifold*. Henceforth, we often neglect to mention the atlas associated with a complex manifold, and we simply say such things as "M is a complex manifold."

Let M and M' be two complex manifolds (possibly of different dimensions) with atlases $\mathcal{A}$ and $\mathcal{A}'$, respectively. A map $F : M \to M'$ is called *complex analytic* if for each chart (V, ψ) in $\mathcal{A}'$, the following property is satisfied: for each chart (U, ϕ) in $\mathcal{A}$ such that $F(U) \subset V$, the map

$$\psi \circ F \circ \phi^{-1} : \phi(U) \longrightarrow \psi(V)$$

is complex analytic. A complex analytic map $F : M \to M'$ is a *conformal equivalence* if F has an inverse function $F^{-1} : M' \to M$ that is also complex analytic, and in such a case, we say that the complex manifolds M and M' are *conformally equivalent*.

In the particular case where the dimension of the complex analytic manifold M is one, we call M a *Riemann surface*. For example, let $\widehat{\mathbb{C}}$ denote the compactification $\mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$. Let $\mathrm{id}_\mathbb{C}$ and J denote the maps

$$\mathrm{id}_\mathbb{C} : \mathbb{C} \longrightarrow \mathbb{C}$$
$$w \mapsto w$$
$$J : \mathbb{C}^\times \cup \{\infty\} \longrightarrow \mathbb{C}$$

$$w \mapsto \frac{1}{w}.$$

Then the collection $\mathcal{A} = \{(\mathbb{C}, \mathrm{id}_{\mathbb{C}}), (\mathbb{C}^\times \cup \{\infty\}, J)\}$ is a complex analytic atlas for $\widehat{\mathbb{C}}$. Henceforth, we assume that the one-dimensional complex analytic manifold $\widehat{\mathbb{C}}$ is equipped with this atlas, and we call this the *standard sphere*.

The standard sphere plays a fundamental role in what follows, mainly due to the following uniformization theorem.

Theorem 5.1.1 (Riemann Uniformization Theorem)**.** *Any compact Riemann surface of genus zero is conformally equivalent to the standard sphere* $\widehat{\mathbb{C}}$*.*

For the purposes of these notes, we omit a detailed proof. We direct the reader to [D] for a more detailed discussion.

As a consequence of the uniformization theorem, we use the term *sphere* to refer to a compact genus-zero Riemann surface. A point p on a sphere together with an element ϵ of $\{+, -\}$ is called an *oriented point* of a sphere.

Let S be a sphere. An *oriented tube* on S is an oriented point p (called a *puncture*) together with a pair (U, ϕ) where U is an open set containing p and $\phi : U \to \mathbb{C}$ is a complex analytic injective map such that $\phi(p) = 0$. Such a pair (U, ϕ) is called a *local coordinate chart vanishing at* p. A *sphere with tubes of type* (m, n) is a sphere S together with m negatively oriented tubes and n positively oriented tubes, whose punctures are all distinct.

If S is a sphere with tubes of type $(1, n)$ then we will denote the negative puncture by p_0 and the n positive punctures by $p_1, \ldots, p_n$. We denote this sphere by

$$(S; p_0, \ldots, p_n; (U_0, \phi_0), \ldots, (U_n, \phi_n))$$

or more succinctly by

$$\mathfrak{S} = \mathfrak{S}(S, n, p, U, \phi).$$

Henceforth, we only concern ourselves with spheres with tubes of type $(1, n)$ for n a nonnegative integer, and any mention of a sphere with tubes should be understood in this way.

Remark 5.1.2. The notion of "tube" given above in terms of punctures and local analytic coordinates vanishing at these punctures may seem divorced from the usual notion from geometry. Nevertheless, the information given by such a "tube" is in fact conformally equivalent to a half-infinite tube in the geometric sense. More precisely, let p be a positively oriented point on a sphere S, and let (U, ϕ) be a local coordinate chart vanishing at p. (This procedure will work similarly for negatively oriented points.) For a complex number z and a positive real number r, denote by B_z^r the open disc of radius r centered at z

$$B_z^r = \{w \in \mathbb{C} : |w - z| < r\}.$$

There is some $r > 0$ so that $B_0^r \subset \phi(U)$. Let T_r denote the set of complex numbers

$$T_r = \{z \in \mathbb{C} : \mathrm{Re}(z) < \log(r)\}.$$

Define an equivalence relation $\sim$ on T_r by $p \sim q$ if and only if $p = q + 2\pi k i$ for some integer k. Let τ_r denote the quotient $\tau_r = T/\sim$ endowed with the quotient topology inherited from the usual metric on T_r. Then τ_r is a half-infinite tube in the usual geometric sense. Moreover, the map

$$\mathrm{Log} \circ \phi : \phi^{-1}(B_0^r) \longrightarrow \tau_r$$

is a conformal equivalence of Riemann surfaces.

The notion of conformal equivalence may be extended to spheres in a natural way. In particular, let

$$\mathfrak{S}_1 = \mathfrak{S}_1(S_1, m, U, p, \phi)$$
$$\mathfrak{S}_2 = \mathfrak{S}_2(S_2, n, V, q, \psi)$$

be two spheres with tubes. We say that these two spheres with tubes are *conformally equivalent* if $m = n$ and if there is a conformal equivalence $F : S_1 \to S_2$ such that

(i) $F(p_i) = q_i$ and

(ii) ϕ_i and $\psi_i \circ F$ are equal when restricted to some neighborhood of p_i.

We denote the conformal equivalence class of a sphere with tubes $\mathfrak{S}$ by $[\mathfrak{S}]$.

Remark 5.1.3. It is apparent from the definition of conformal equivalence that the conformal data of a sphere with tubes $\mathfrak{S}_1(S_1, m, U, p, \phi)$ are given by the following:

- the surface S,
- the number of positive punctures m,
- the locations of all $m + 1$ punctures, and
- the *germs* of the local coordinate maps ϕ_i.

Indeed, condition (ii) asserts that a conformal equivalence $F : S_1 \to S_2$ must only respect the germs of the local coordinate maps—not the maps themselves.

For a positive integer n, the symmetric group S_n acts on the set of all spheres with tubes of type $(1, n)$ in a natural way. Specifically, we define a *left* action

$$\begin{aligned} S_n \times \{\text{spheres with tubes of type } (1, n)\} &\longrightarrow \{\text{spheres with tubes of type } (1, n)\} \\ (\sigma, \mathfrak{S}) &\mapsto \sigma(\mathfrak{S}) \end{aligned}$$

of S_n on the set of all spheres with tubes of type $(1, n)$ in the following way: if σ is an element of S_n and if $\mathfrak{S}$ is a sphere with tubes of type $(1, n)$, then let $\sigma(\mathfrak{S})$ denote the sphere with tubes of type $(1, n)$ where the i-th incoming puncture of $\sigma(\mathfrak{S})$ is the $\sigma^{-1}(i)$-th incoming puncture of $\mathfrak{S}$ for each $i = 1, \ldots, n$.

Remark 5.1.4. This action simply amounts to a reordering of the punctures, and thus is conformally invariant, in the sense that it requires as input data only items found in Remark 5.1.3. This implies that we obtain a well-defined left group action of S_n on the space of conformal equivalence classes of spheres with tubes of type $(1, n)$, which we denote by

$$(\sigma, [\mathfrak{S}]) \mapsto \sigma([\mathfrak{S}]).$$

5.2 The sewing operation

In this section, we follow Huang [H97] to introduce a "sewing operation" for spheres with tubes. The idea behind sewing will be to "sew" the 0-th tube of one sphere with tubes into the i-th tube of another sphere with tubes.

For a positive real number r, denote by B^r_∞ the open disc of radius r about ∞

$$B^r_\infty = \{w \in \widehat{\mathbb{C}} : |1/w| < r\}.$$

Denote by $\mathrm{Cl}(B^r_z)$ the closed disc about $z \in \mathbb{C}$

$$\mathrm{Cl}(B^r_z) = \{w \in \mathbb{C} : |w - z| \leq r\},$$

and similarly by $\mathrm{Cl}(B^r_\infty)$ the closed disc about ∞

$$\mathrm{Cl}(B^r_\infty) = \{w \in \widehat{\mathbb{C}} : |1/w| \leq r\}.$$

Let

$$\begin{aligned} \mathfrak{S}_1 &= \mathfrak{S}_1(S_1, m, U, p, \phi) \\ \mathfrak{S}_2 &= \mathfrak{S}_2(S_2, n, V, q, \psi) \end{aligned}$$

be two spheres with tubes with $m > 0$. Let i be an integer satisfying $1 \leq i \leq m$. Assume that there is a positive number r such that

$$\mathrm{Cl}(B_0^r) \subset \phi_i(U_i)$$
$$\mathrm{Cl}(B_0^{1/r}) \subset \psi_0(V_0)$$

and that p_i and q_0 are the only punctures in $\phi_i^{-1}(\mathrm{Cl}(B_0^r))$ and $\psi_0^{-1}(\mathrm{Cl}(B_0^{1/r}))$, respectively. Then we say that *the i-th tube of* $\mathfrak{S}_1$ *can be sewn with the* 0*-th tube of* $\mathfrak{S}_2$.

If the i-th tube of $\mathfrak{S}_1$ can be sewn with the 0-th tube of $\mathfrak{S}_2$, then we have the following *sewing procedure*.

There are real numbers r_1 and r_2 satisfying $0 < r_2 < r < r_1$ such that

$$\mathrm{Cl}(B_0^{r_1}) \subset \phi_i(U_i)$$
$$\mathrm{Cl}(B_0^{1/r_2}) \subset \psi_0(V_0).$$

Define an equivalence relation on the disjoint union

$$(S_1 \setminus \phi_i^{-1}(\mathrm{Cl}(B_0^{r_2}))) \sqcup (S_2 \setminus \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_1})))$$

by $p \sim q$ if an only if one of the following two conditions is satisfied

- $p = q$ or
- $p \in \phi_i^{-1}(\mathrm{Cl}(B_0^{r_1})) \setminus \phi_i^{-1}(\mathrm{Cl}(B_0^{r_2}))$ and $q \in \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_2})) \setminus \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_1}))$ satisfy

$$(\psi_0^{-1} \circ J \circ \phi_i)(p) = q. \tag{5.2.1}$$

Let $S_1\ {}_i\infty_0\ S_2$ denote the one-dimensional complex manifold given by the topological space

$$((S_1 \setminus \phi_i^{-1}(\mathrm{Cl}(B_0^{r_2}))) \sqcup (S_2 \setminus \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_1}))))/ \sim$$

together with the atlas determined by the atlases of $S_1 \setminus \phi_i^{-1}(\mathrm{Cl}(B_0^{r_2}))$ and $S_2 \setminus \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_1}))$ with the complex analytic transition map

$$\psi_0^{-1} \circ J \circ \phi_i : \phi_i^{-1}(\mathrm{Cl}(B_0^{r_1})) \setminus \phi_i^{-1}(\mathrm{Cl}(B_0^{r_2})) \longrightarrow \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_2})) \setminus \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_1})).$$

The punctures with ordering on $S_1\ {}_i\infty_0\ S_2$ are given by

$$p_0, p_1, \ldots, p_{i-1}, q_1, q_2, \ldots, q_n, p_{i+1}, p_{i+2}, \ldots, p_m.$$

For $j \neq i$, let

$$U_j' = U_j \setminus \phi_i^{-1}(\mathrm{Cl}(B_0^r))$$

$$\phi_j' = \phi_j|_{U_j'},$$

and for $k \neq 0$, let

$$V_k' = V_k \setminus \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r}))$$
$$\psi_k' = \psi_k|_{V_k'}.$$

Then the tubes centered at the above ordered punctures are

$$(U_0', \phi_0'), \ldots, (U_{i-1}', \phi_{i-1}'), (V_1', \psi_1'), \ldots, (V_n', \psi_n'), (U_{i+1}', \phi_{i+1}'), \ldots, (U_m', \phi_m').$$

We let $\mathfrak{S}_1 \,{}_i\infty_0\, \mathfrak{S}_2$ denote the resulting sphere with $m + n - 1$ tubes as described above.

Remark 5.2.1. The conformal equivalence class of the sewn sphere with tubes $\mathfrak{S}_1 \,{}_i\infty_0\, \mathfrak{S}_2$ is determined by the following input data:

- the surfaces S_1 and S_2,
- the number of positive punctures m and n,
- the locations of all $m + 1$ punctures on S_1 and the location of all $n + 1$ punctures on S_2, and
- the germs of the local coordinate maps ϕ_i and ψ_j.

These data coincide with the conformal data of the spheres with tubes $\mathfrak{S}_1$ and $\mathfrak{S}_2$, and hence the sewing operation gives rise to a well-defined operation on conformal equivalence classes of spheres with tubes.

5.3 The moduli space of spheres with tubes

Following Huang [H97], we introduce more explicit descriptions of the collections of all conformal equivalence classes of spheres with tubes of type $(1, n)$ for n a nonnegative integer.

The collection of all equivalence classes of spheres with tubes of type $(1, n)$ is called the *moduli space of spheres with tubes of type* $(1, n)$, and we denote this moduli space by $\mathcal{K}(n)$. The collection of all equivalence classes of spheres with tubes is called the *moduli space of spheres with tubes*, and we denote this moduli space by

$$\mathcal{K} = \bigcup_{n \in \mathbb{N}} \mathcal{K}(n).$$

The moduli space of spheres with tubes of type $(1, n)$ will be realized as a subset of the set $\mathbb{C}^\infty$ of infinite sequences of complex numbers. This description allows one to introduce a notion of a meromorphic map from the moduli space of spheres with tubes of type $(1, n)$ as a certain type of rational function in variables given by the coordinates of these sequences of complex numbers.

Proposition 5.3.1. *Let n be a positive integer and let $\mathfrak{S} = \mathfrak{S}(S, n, p, U, \phi)$ be a sphere with tubes of type $(1, n)$. Then $\mathfrak{S}$ is conformally equivalent to a sphere with tubes of the form*

$$(\widehat{\mathbb{C}}; \infty, z_1, \ldots, z_{n-1}, 0; (B_\infty^{r_0}, \psi_0), (B_{z_1}^{r_1}, \psi_1), \ldots, B_0^{r_n}, \psi_n)) \tag{5.3.1}$$

where $z_1, \ldots, z_{n-1}$ are nonzero distinct complex numbers and $\psi_0, \ldots, \psi_n$ are analytic functions such that

$$\begin{aligned} \psi_0(\infty) &= 0 \\ \psi_i(z_i) &= 0 \quad \textit{for each } i = 1, \ldots, n-1 \\ \psi_n(0) &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{w \to \infty} w\psi_0(w) &= 1 \\ \lim_{w \to z_i} \frac{\psi_i(w)}{w - z_i} &\neq 0 \quad \textit{for each } i = 1, \ldots, n-1 \\ \lim_{w \to 0} \frac{\psi_n(w)}{w} &\neq 0. \end{aligned}$$

Proof. By the Riemann Uniformization Theorem 5.1.1, there is a complex analytic isomorphism $F : S \to \widehat{\mathbb{C}}$, giving a sphere with tubes

$$\mathfrak{S}_1 = (\widehat{\mathbb{C}}; F(p_0), \ldots, F(p_n), (F(U_0), \phi_0 \circ F^{-1}), \ldots, (F(U_n), \phi_n \circ F^{-1}))$$

that is conformally equivalent to $\mathfrak{S}$. The goal is to now find an automorphism T of the standard sphere which

(i) sends $F(p_0)$ to ∞,

(ii) sends $F(p_n)$ to 0,

(iii) satisfies the property that when $(\phi_0 \circ F^{-1} \circ T^{-1})(w)$ is expanded about ∞, the coefficient of w^{-1} is 1.

Define complex numbers $a_1^{(0)}$ and A by

$$a_1^{(0)} = \lim_{w \to F(p_0)} \frac{(\phi_0 \circ F^{-1})(w)}{w - F(p_0)}$$

$$A = a_1^{(0)}(F(p_0) - F(p_n)).$$

Let $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the conformal map

$$T(w) = \frac{1}{A} \cdot \frac{w - F(p_n)}{w - F(p_0)}.$$

Let $\mathfrak{S}_2$ denote the sphere with tubes given by

$$\mathfrak{S}_2 = (\widehat{\mathbb{C}}; \infty, z_1, \dots, z_{n-1}, 0; ((T \circ F)(U_0), \phi_0 \circ F^{-1} \circ T^{-1}), \\ \dots, ((T \circ F)(U_n), \phi_n \circ F^{-1} \circ T^{-1}))$$

where $z_i = (T \circ F)(p_i)$ for each $i = 1, \dots, n-1$. Then $\mathfrak{S}_2$ is conformally equivalent to $\mathfrak{S}_1$ by the map T (and hence also conformally equivalent to $\mathfrak{S}$ by the map F). Choose positive real numbers $r_0, \dots, r_n$ such that

$$\begin{aligned} &B_\infty^{r_0} \subset (T \circ F)(U_0), \\ &B_{z_i}^{r_i} \subset (T \circ F)(U_i), \qquad \text{for each } i = 1, \dots, n-1, \\ &B_0^{r_n} \subset (T \circ F)(U_n). \end{aligned}$$

Let

$$\begin{aligned} \psi_0 &= \phi_0 \circ F^{-1} \circ T^{-1}|_{B_\infty^{r_0}}, \\ \psi_i &= \phi_i \circ F^{-1} \circ T^{-1}|_{B_{z_i}^{r_i}} \qquad \text{for each } i = 1, \dots, n-1 \\ \psi_n &= \phi_n \circ F^{-1} \circ T^{-1}|_{B_0^{r_n}}. \end{aligned}$$

Then $\mathfrak{S}_2$ is conformally equivalent to the sphere with tubes given by

$$(\widehat{\mathbb{C}}; \infty, z_1, \dots, z_{n-1}, 0; (B_\infty^{r_0}, \psi_0), (B_{z_1}^{r_1}, \psi_1), \dots, B_0^{r_n}, \psi_n)).$$

Moreover, we have

$$\begin{aligned} \lim_{w \to \infty} w\psi_0(w) &= \lim_{w \to \infty} w\phi_0 \circ F^{-1} \circ T^{-1}|_{B_\infty^{r_0}}(w) \\ &= \lim_{u \to F(p_0)} T(u)(\phi_0 \circ F^{-1}(u)) \\ &= \lim_{u \to F(p_0)} \frac{1}{A}(u - F(p_n)) \frac{\phi_0 \circ F^{-1}(u)}{u - F(p_0)} \\ &= \lim_{u \to F(p_0)} \frac{1}{a_1^{(0)}(F(p_0) - F(p_n))}(u - F(p_n)) a_1^{(0)} \\ &= 1. \end{aligned}$$

The fact that

$$\lim_{w \to z_i} \frac{\psi_i(w)}{w - z_i} \neq 0 \quad \text{for each } i = 1, \ldots, n-1$$

follows from the observation that $\psi_i = \phi_i \circ F^{-1} \circ T^{-1}|_{B^{r_i}_{z_i}}$ is injective on a neighborhood of z_i. Similarly, we obtain

$$\lim_{w \to 0} \frac{\psi_n(w)}{w} \neq 0,$$

as desired. □

For n a positive integer, a sphere of the form of (5.3.1) is called a *canonical sphere of tubes of type* $(1, n)$. The next proposition describes when two canonical spheres with tubes are conformally equivalent. We must first recall the following well-known result from complex analysis.

Lemma 5.3.2. *The set of all conformal equivalences from the standard sphere $\widehat{\mathbb{C}}$ to itself is the set of all Möbius transformations, that is, the set of all rational functions $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the form*

$$T(w) = \frac{aw + b}{cw + d}$$

where $ad - bc \neq 0$.

Proposition 5.3.3. *Let n be a positive integer, and let*

$$\mathfrak{S} = (\widehat{\mathbb{C}}; \infty, z_1, \ldots, z_{n-1}, 0; (B^{r_0}_{\infty}, \psi_0), (B^{r_1}_{z_1}, \psi_1), \ldots, B^{r_n}_0, \psi_n))$$

and

$$\widetilde{\mathfrak{S}} = (\widehat{\mathbb{C}}; \infty, \widetilde{z_1}, \ldots, \widetilde{z_{n-1}}, 0; (B^{\widetilde{r_0}}_{\infty}, \widetilde{\psi_0}), (B^{\widetilde{r_1}}_{\widetilde{z_1}}, \widetilde{\psi_1}), \ldots, B^{\widetilde{r_n}}_0, \widetilde{\psi_n}))$$

be two canonical spheres with tubes of type $(1, n)$. Then $\mathfrak{S}$ and $\widetilde{\mathfrak{S}}$ are conformally equivalent if and only if $z_i = \widetilde{z_i}$ for each $i = 1, \ldots, n-1$ and $\psi_i \equiv \widetilde{\psi_i}$ on some neighborhood of $z_i = \widetilde{z_i}$ for each $i = 0, \ldots, n$.

Proof. If $z_i = \widetilde{z_i}$ for each $i = 1, \ldots, n-1$ and $\psi_i = \widetilde{\psi_i}$ for each $i = 0, \ldots, n$, then it is clear that the identity map will give a conformal equivalence between $\mathfrak{S}$ and $\widetilde{\mathfrak{S}}$.

On the other hand, suppose that $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a conformal equivalence from $\mathfrak{S}$ onto $\widetilde{\mathfrak{S}}$. Then by definition we have

$$F(\infty) = \infty$$
$$F(0) = 0.$$

Therefore, since F is a Möbius transformation, we see that $F(w) = aw$ for some $a \neq 0$. Since $\widetilde{\mathfrak{S}}$

is a standard sphere, we know that $\widetilde{\psi_0}$ satisfies

$$\lim_{w \to \infty} w\widetilde{\psi_0}(w) = 1.$$

Also, since F is a conformal equivalence, we have

$$\widetilde{\psi}_0|_{B_\infty^{\min(r_0,\widetilde{r_0})}} \equiv \psi_0 \circ F^{-1}|_{B_\infty^{\min(r_0,\widetilde{r_0})}}.$$

Therefore, we see that

$$\lim_{w \to \infty} w\psi_0\left(\frac{w}{a}\right) = 1.$$

Since $\mathfrak{S}$ is also a canonical sphere, the function ψ_0 may be expanded in the form

$$\psi_0(w) = w^{-1} + \sum_{j \in \mathbb{Z}_+} a_j w^{-j-1}.$$

This implies that $a = 1$, and therefore F is the identity map. The proposition now follows. $\square$

Therefore, we see that the conformal equivalence data of a canonical sphere consist only of the complex numbers z_i and the germs of the local analytic coordinates ψ_i. Expanding these coordinates $\{\psi_i\}_{i=0}^n$ in terms of their unique Laurent series expansions about ∞, z_i, or 0, respectively, we obtain, for each coordinate ψ_i, an infinite list of complex numbers, namely, the coefficients of its Laurent expansion. Thus, for each positive integer n, we have made a bijective correspondence

$$\mathcal{K}(n) \longleftrightarrow T(n) \subset \mathbb{C}^\infty \tag{5.3.2}$$

where $T(n)$ is some subset of the space of infinite countable sequences with coefficients in $\mathbb{C}$. It remains to consider the moduli space $\mathcal{K}(0)$. In this case, we will again obtain an identification similar to the one above. The only difference will be more restrictive conditions on the sequence of coefficients associated with the local coordinate vanishing at ∞. These extra conditions come from the fact that such tubes lack a second puncture, which would usually be sent to zero using a Möbius transformation. This extra degree of freedom in the Möbius transformation can instead be used to place another condition on the local coordinate vanishing at ∞, namely that the coefficient $a_2^{(0)}$ is 0. More specifically, we have the following results, which can be proved in manners similar to Propositions 5.3.1 and 5.3.3.

Proposition 5.3.4. *Any sphere with tubes of type* $(1,0)$ *is conformally equivalent to a sphere with tubes of the form*

$$(\widehat{\mathbb{C}}; \infty; (B_\infty^{r_0}, \psi_0))$$

where ψ_0 *may be expanded as*

$$\psi_0(w) = \frac{1}{w} + \sum_{j\in\mathbb{Z}_+ + 1} a_j w^{-j-1}. \tag{5.3.3}$$

A sphere with tubes of type $(1,0)$ as in Proposition 5.3.4 is called a *canonical sphere with tubes of type* $(1,0)$.

Proposition 5.3.5. *Let*

$$\mathfrak{S} = (\widehat{\mathbb{C}}; \infty; (B_\infty^{r_0}, \psi_0))$$

and

$$\widetilde{\mathfrak{S}} = (\widehat{\mathbb{C}}; \infty; (B_\infty^{\tilde{r}_0}, \widetilde{\psi_0}))$$

be two canonical spheres with tubes of type $(1,0)$*. Then* $\mathfrak{S}$ *and* $\widetilde{\mathfrak{S}}$ *are conformally equivalent if and only if* $\psi_0 \equiv \widetilde{\psi_0}$ *on some neighborhood of* ∞. □

As a consequence of Propositions 5.3.1, 5.3.3, 5.3.4, and 5.3.5, we have the following corollary, which makes more explicit the identification in (5.3.2).

Corollary 5.3.6. *Let* n *be a positive integer. Then the moduli space of spheres with tubes of type* $(1,n)$ *can be identified with the set of all* $2n$*-tuples of the form*

$$(z_1, \ldots, z_{n-1}; \psi_0, \ldots, \psi_n)$$

where z_i *are distinct nonzero complex numbers, the functions* ψ_i *have Laurent series expansions about* ∞, z_i*, and* 0*, respectively, of the form*

$$\psi_0(w) = w^{-1} + \sum_{j\in\mathbb{Z}_+} a_j^{(0)} w^{-j-1}$$
$$\psi_i(w) = \sum_{j\in\mathbb{Z}_+} a_j^{(i)} (w - z_i)^j \qquad \text{for } 1 \le i \le n-1$$
$$\psi_n(w) = \sum_{j\in\mathbb{Z}_+} a_j^{(n)} w^j$$

for some complex numbers $a_j^{(i)} \in \mathbb{C}$ *such that* $a_0^{(i)} \neq 0$ *for each* $i = 1, \ldots, n$*, and these Laurent series are absolutely convergent in neighborhoods of* $w = \infty$*,* $w_i = z_i$ *for* $1 \le i \le n-1$*, and* $w = 0$*, respectively. The moduli space of spheres with tubes of type* $(1,0)$ *can be identified with the set of all power seres* ψ_0 *of the form* (5.3.3) *that are absolutely convergent in a neighborhood of* ∞. □

The correspondence of Corollary 5.3.6 allows us to define the notion of a meromorphic function on the moduli space of spheres with tubes in terms of a rational function in certain variables with restrictions on the poles. More specifically, for a positive integer n, the moduli space of spheres with tubes of type $(1, n)$ may be identified with a subset of the set of sequences of the form

$$\{(z_1, \ldots, z_{n-1}), \{a_j^{(i)}\}\}$$

where $\{z_i : 1 \leq i \leq n-1\}$ are distinct nonzero complex numbers and $\{a_j^{(i)} : j \in \mathbb{N}, 0 \leq i \leq n\}$ are complex numbers such that

$$a_0^{(i)} \neq 0 \quad \text{for each } i = 1, \ldots, n.$$

A *meromorphic function on the moduli space of spheres with tubes of type* $(1, n)$ is a rational function of the form

$$\frac{P(\{z_i\}, \{a_j^{(i)}\})}{\left(\prod_{i=1}^{n}(a_0^{(i)})^{r_i}\right)\left(\prod_{j=1}^{n-1} z_j^{s_j}\right)\left(\prod_{1\leq k<l\leq n-1}(z_k - z_l)^{t_{kl}}\right)}$$

where $P(\{z_i\}, \{a_j^{(i)}\})$ is a polynomial in the variables $\{z_i : 1 \leq i \leq n-1\}$ and $\{a_j^{(i)} : j \in \mathbb{N}, 0 \leq i \leq n\}$ and r_i, s_j, t_{kl} are nonnegative integers. A *meromorphic function on the moduli space of spheres with tubes of type* $(1, 0)$ is a polynomial in the variables $\{a_j : j \in \mathbb{Z}_+ + 1\}$.

5.4 The sewing equation

Let m be a positive integer and n a nonnegative integer. Let

$$\mathfrak{C}_1 = \mathfrak{C}_1(\widehat{\mathbb{C}}, m, U, p, \phi)$$
$$\mathfrak{C}_2 = \mathfrak{C}_2(\widehat{\mathbb{C}}, n, V, q, \psi)$$

be two canonical spheres with tubes. Suppose that we may sew the 0-th tube of $\mathfrak{C}_2$ to the i-th tube of $\mathfrak{C}_1$. Then the resulting sewn sphere $\mathfrak{C}_1 \,{}_i\infty_0\, \mathfrak{C}_2$ with tubes is conformally equivalent to some canonical sphere with tubes

$$\mathfrak{C}_3 = \mathfrak{C}_3(\widehat{\mathbb{C}}, m+n-1, W, r, \theta),$$

which we describe now.

By the definition of sewing, there are real numbers r_1 and r_2 satisfying $0 < r_2 < r < r_1$ such that

$$\mathrm{Cl}(B_0^{r_1}) \subset \phi_i(U_i)$$
$$\mathrm{Cl}(B_0^{1/r_2}) \subset \psi_0(V_0).$$

Define an equivalence relation on the disjoint union

$$(\widehat{\mathbb{C}} \setminus \phi_i^{-1}(\mathrm{Cl}(B_0^{r_2}))) \sqcup (\widehat{\mathbb{C}} \setminus \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_1})))$$

by $p \sim q$ if an only if one of the following two conditions is satisfied

- $p = q$ or
- $p \in \phi_i^{-1}(\mathrm{Cl}(B_0^{r_1})) \setminus \phi_i^{-1}(\mathrm{Cl}(B_0^{r_2}))$ and $q \in \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_2})) \setminus \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_1}))$ satisfy

$$(\psi_0^{-1} \circ J \circ \phi_i)(p) = q.$$

Let M denote the one-dimensional complex manifold given by the topological space

$$((\widehat{\mathbb{C}} \setminus \phi_i^{-1}(\mathrm{Cl}(B_0^{r_2}))) \sqcup (\widehat{\mathbb{C}} \setminus \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_1}))))/\sim$$

together with the atlas determined by the atlases of $S_1 \setminus \phi_i^{-1}(\mathrm{Cl}(B_0^{r_2}))$ and $S_2 \setminus \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_1}))$ with the complex analytic transition map

$$\psi_0^{-1} \circ J \circ \phi_i : \phi_i^{-1}(\mathrm{Cl}(B_0^{r_1})) \setminus \phi_i^{-1}(\mathrm{Cl}(B_0^{r_2})) \longrightarrow \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_2})) \setminus \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_1})).$$

The sphere with tubes $\mathfrak{C}_1 \, {}_i\infty_0 \, \mathfrak{C}_2$ is the sphere M together with the punctures

$$p_0, p_1, \ldots, p_{i-1}, q_1, q_2, \ldots, q_n, p_{i+1}, p_{i+2}, \ldots, p_m.$$

and tubes given by

$$(U_0', \phi_0'), \ldots, (U_{i-1}', \phi_{i-1}'), (V_1', \psi_1'), \ldots, (V_n', \psi_n'), (U_{i+1}', \phi_{i+1}'), \ldots, (U_m', \phi_m'),$$

as described in Section 5.2.

Let $F : M \to \widehat{\mathbb{C}}$ denote the conformal equivalence from $\mathfrak{C}_1 \, {}_i\infty_0 \, \mathfrak{C}_2$ to the canonical representative $\mathfrak{C}_3$. We call F the *uniformizing function*. By the complex structure we have taken on the sewn sphere, we may express F as two functions, one $F^{(1)}$ from $\widehat{\mathbb{C}} \setminus \phi_i^{-1}(\mathrm{Cl}(B_0^{r_2}))$ to $\widehat{\mathbb{C}}$ and another $F^{(2)}$ from $\widehat{\mathbb{C}} \setminus \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_1}))$ to $\widehat{\mathbb{C}}$ such that

$$F(p) = \begin{cases} F^{(1)}(w), & w \in \widehat{\mathbb{C}} \setminus \phi_i^{-1}(\mathrm{Cl}(B_0^{r_2})) \\ F^{(2)}(w), & w \in \widehat{\mathbb{C}} \setminus \psi_0^{-1}(\mathrm{Cl}(B_0^{1/r_1})) \end{cases}.$$

Moreover, from the sewing operation, we obtain the following conditions on $F^{(1)}$ and $F^{(2)}$.

(i) From the equivalence relation on the overlapping annulus given by (5.2.1), we find that if

$w \in \phi_i^{-1}(\mathrm{Cl}(B_0^{r_1})) \setminus \phi_i^{-1}(\mathrm{Cl}(B_0^{r_2}))$, then

$$((F^{(2)})^{-1} \circ F^{(1)})(w) = (\psi_0^{-1} \circ J \circ \phi_i)(w). \tag{5.4.1}$$

Equation (5.4.1) is called the *sewing equation*.

(ii) In order for the sphere with tubes $\mathfrak{C}_3$ to be canonical, we require some extra conditions on the uniformizing function F. In particular, we need the first puncture of the resulting sphere $\mathfrak{C}_3$ to be at infinity and the last puncture to be located at zero—if such a puncture exists. We also require that the local coordinates at these punctures satisfy the conditions of Proposition 5.3.1. In the particular case where $i = m$ and $n \neq 0$, these conditions are equivalent to the conditions

$$F^{(1)}(\infty) = \infty \tag{5.4.2}$$

$$F^{(2)}(0) = 0 \tag{5.4.3}$$

$$\lim_{w \to \infty} \frac{1}{w} F^{(1)}(w) = 1, \tag{5.4.4}$$

which are called the *normalization conditions* for the function F.

In the case where $i = m$ and $n \neq 0$, conditions (i) and (ii) completely determine a uniformizing function F, and the resulting sphere with tubes $\mathfrak{C}_3$ is a canonical sphere with tubes of type $(1, m + n - 1)$.

In the case where $i \neq m$ or $n = 0$, one can still find an F satisfying conditions (i) and (ii) to map the sewn sphere with tubes $\mathfrak{C}_1 \, {}_i\infty_0 \, \mathfrak{C}_2$ to some sphere with tubes $\mathfrak{C}_3$, but the resulting sphere with tubes may not be canonical. If $n = 0$ and $i = m$ for $m > 1$, or if $m \neq n$, then the last puncture of the resulting sphere with tubes will not be at zero, since F sends the zero puncture of the second sphere to zero. If $n = 0$ and $m = 1$, then the resulting sphere will have only one puncture (at infinity), and the local coordinate at this puncture may not satisfy the extra condition of Proposition 5.3.4, namely that the second coefficient of the power series expansion be zero. In these cases, to find a canonical sphere with tubes conformally equivalent to $\mathfrak{C}_3$, we simply compose F with a desired Möbius transformation, as in the proof of Proposition 5.3.1.

5.5 A formal study of the right hand side of the sewing equation

The goal of this section is to explicitly describe a solution F to the sewing equation (5.4.1) of the previous section. We follow the treatment presented in Huang [H97], highlighting the major points while skipping over some of the finer details.

The following result from [H97] gives an equivalent way of expanding formal power series.

Theorem 5.5.1. *Let R be a commutative algebra over $\mathbb{Q}$ with identity. Let*

$$f(w) = a_0 \left(w + \sum_{n\in\mathbb{Z}_+} a_j w^{j+1} \right)$$

be a formal power series in $R[[w]]$. Then there is a unique sequence $A = \{A_j\}_{j=1}^{\infty}$ in R^{∞} such that

$$f(w) = \exp\left(\sum_{j\in\mathbb{Z}_+} A_j w^{j+1} \frac{d}{dw} \right) a_0^{w\frac{d}{dw}} w,$$

where $a_0^{w\frac{d}{dw}}$ is the linear operator on $R[[w, w^{-1}]]$ defined by

$$a_0^{w\frac{d}{dw}} w^n = a_0^n w^n$$

for $n \in \mathbb{Z}$.

The following result is a particular case of a more general result taken from [Ba03], where it is shown how to "compose" two power series in the exponential form of Theorem 5.5.1.

Theorem 5.5.2. *Let R be a commutative algebra over $\mathbb{Q}$ with identity. Let $g(w) \in R[[w, w^{-1}]]$ and let $f(w) \in R[[w]]$. Write*

$$f(w) = \exp\left(\sum_{j\in\mathbb{N}} A_j w^{j+1} \frac{d}{dw} \right) a_0^{w\frac{d}{dw}} w,$$

for some $A_j \in R$ and $a_0 \in R$. If $(g \circ f)(w)$ is a well-defined element of $R[[w, w^{-1}]]$, then

$$(g \circ f)(w) = \exp\left(\sum_{j\in\mathbb{N}} A_j w^{j+1} \frac{d}{dw} \right) a_0^{w\frac{d}{dw}} \cdot g(w).$$

Moreover, we need the following proposition, found in [H97], on how to "invert" power series.

Theorem 5.5.3. *Let R be a commutative algebra over $\mathbb{Q}$ with identity. Let*

$$f(w) = a_0 \left(w + \sum_{n\in\mathbb{Z}_+} a_j w^{j+1} \right)$$

be a formal power series in $R[[w]]$, and write

$$f(w) = \exp\left(\sum_{j\in\mathbb{Z}_+} A_j w^{j+1} \frac{d}{dw} \right) a_0^{w\frac{d}{dw}} w$$

for some unique sequence $\{A_j\}_{j=1}^{\infty}$ in R^{∞}. Suppose that a_0 is an invertible element of R. Then there is a unique formal series $f^{-1}(w) \in R[[w, w^{-1}]]$ such that

$$(f \circ f^{-1})(w) = (f^{-1} \circ f)(w).$$

Moreover, the series $f^{-1}(w) \in R[[w, w^{-1}]]$ is given by

$$f^{-1}(w) = a_0^{-w\frac{d}{dw}} \exp\left(-\sum_{j\in\mathbb{Z}_+} A_j w^{j+1}\frac{d}{dw}\right) w.$$

We now study the local coordinate maps vanishing at ∞ as formal power series. Let

$$g(z) = \frac{1}{z} + \sum_{j\in\mathbb{Z}_+} b_j \left(\frac{1}{z}\right)^{j+1}$$

be a formal series in $z^{-1}R[[z^{-1}]]$. Then $g_{\#}(z) = g(\frac{1}{z})$ is a formal power series in $wR[[w]]$. By Theorem 5.5.1, there is a unique sequence $\{B_j\}_{j=1}^{\infty}$ in R^{∞} such that

$$g_{\#}(z) = \exp\left(\sum_{j\in\mathbb{Z}_+} B_j z^{j+1}\frac{d}{dz}\right) z.$$

Now let $z = \frac{1}{w}$. Then note that

$$dz = -\frac{dw}{w^2}$$

and hence

$$\frac{d}{dz} = -w^2\frac{d}{dw}.$$

It now follows that

$$\begin{aligned}
g(w) &= g_{\#}\left(\frac{1}{w}\right) \\
&= g_{\#}(z) \\
&= \exp\left(\sum_{j\in\mathbb{Z}_+} B_j z^{j+1}\frac{d}{dz}\right) z \\
&= \exp\left(\sum_{j\in\mathbb{Z}_+} B_j w^{-j-1}\left(-w^2\frac{d}{dw}\right)\right)\frac{1}{w} \\
&= \exp\left(-\sum_{j\in\mathbb{Z}_+} B_j w^{-j+1}\frac{d}{dw}\right)\frac{1}{w}.
\end{aligned}$$

To summarize, we have the following proposition.

Proposition 5.5.4. *Let*

$$g(z) = \frac{1}{z} + \sum_{j \in \mathbb{Z}_+} b_j \left(\frac{1}{z}\right)^{j+1}$$

be a formal series in $z^{-1}R[[z^{-1}]]$. Then there is a unique sequence $\{B_j\}_{j=1}^{\infty}$ in R^{∞} such that

$$g(w) = \exp\left(-\sum_{j \in \mathbb{Z}_+} B_j w^{-j+1} \frac{d}{dw}\right) \frac{1}{w}.$$

Moreover, we may find the inverse for g in the following manner.

Proposition 5.5.5. *Let*

$$g(z) = \frac{1}{z} + \sum_{j \in \mathbb{Z}_+} b_j \left(\frac{1}{z}\right)^{j+1}$$

be a formal series in $z^{-1}R[[z^{-1}]]$. Then g has an inverse in $R[[z, z^{-1}]]$ given by

$$g^{-1}(w) = \frac{1}{g_{\#}^{-1}(w)}.$$

Proof. By Theorem 5.5.3, the formal series $g_{\#}(w)$ has in inverse $g_{\#}^{-1}(w)$. It remains to show that the inverse for g is given by the formula

$$g^{-1}(w) = \frac{1}{g_{\#}^{-1}(w)}.$$

But this is the case, for

$$(g \circ g^{-1})(w) = g\left(\frac{1}{g_{\#}^{-1}(w)}\right) = g\left(\frac{1}{g_{\#}^{-1}(w)}\right) = g_{\#}(g_{\#}^{-1}(w)) = w$$

and

$$(g^{-1} \circ g)(w) = \frac{1}{g_{\#}^{-1}(g(w))} = \frac{1}{g_{\#}^{-1}\left(g_{\#}\left(\frac{1}{w}\right)\right)} = \frac{1}{\frac{1}{w}} = w,$$

as desired. □

Finally, we need to know how to "push" the operator $a_0^{w\frac{d}{w}}$ past exponentials, as described in the next proposition.

Proposition 5.5.6. *Let R be a commutative algebra over $\mathbb{Q}$ with identity. Then the equality*

$$\exp\left(\sum_{j\in\mathbb{Z}_+} C_j w^{-j+1}\frac{d}{dw}\right) a_0^{w\frac{d}{dw}} w = a_0^{w\frac{d}{dw}} \exp\left(\sum_{j\in\mathbb{Z}_+} B_j w^{-j+1}\frac{d}{dw}\right) w \tag{5.5.1}$$

holds in $R[[w, w^{-1}]]$ if and only if $C_j = a_0^{-j} B_j$ for each j.

Proof. Let

$$f(w) = a_0^{w\frac{d}{dw}} w$$
$$g(w) = \exp\left(\sum_{j\in\mathbb{Z}_+} B_j w^{-j+1}\frac{d}{dw}\right) w.$$

Then Theorem 5.5.2 implies that (5.5.1) is equivalent to

$$\exp\left(\sum_{j\in\mathbb{Z}_+} C_j w^{-j+1}\frac{d}{dw}\right) a_0^{w\frac{d}{dw}} w = (g\circ f)(w) = g(a_0 w),$$

that is,

$$\exp\left(\sum_{j\in\mathbb{Z}_+} C_j w^{-j+1}\frac{d}{dw}\right)\cdot(a_0 w) = \exp\left(\sum_{j\in\mathbb{Z}_+} B_j a_0^{-j} w^{-j+1}\frac{d}{dw}\right)\cdot(a_0 w).$$

The result now follows. □

As a consequence of these results, we may write the right hand side of the sewing equation (5.4.1) in another form.

We may express the local coordinates $\phi_i(w)$ and $\psi_0(w)$ as power series, and we may regard them as formal power series in $\mathbb{C}[[w]]$ and $\mathbb{C}[[w^{-1}]]$ respectively. By Theorem 5.5.1, there is a sequence $\{A_j\}_{j=1}^{\infty}$ in $\mathbb{C}^{\infty}$ such that

$$\phi_i(w) = \exp\left(\sum_{j\in\mathbb{Z}_+} A_j w^{j+1}\frac{d}{dw}\right) a_0^{w\frac{d}{dw}} w,$$

and by Proposition 5.5.4, there is a sequence $\{B_j\}_{j=1}^{\infty}$ in $\mathbb{C}^{\infty}$ such that

$$\psi_0(w) = \exp\left(-\sum_{n\in\mathbb{Z}_+} B_j w^{-j+1}\frac{d}{dw}\right)\frac{1}{w}.$$

Let

$$(\psi_0)_\#(w) = \psi_0\left(\frac{1}{w}\right) = \exp\left(\sum_{j\in\mathbb{Z}_+} B_j w^{j+1}\frac{d}{dw}\right) w.$$

Applying Proposition 5.5.5, we find that the inverse of ψ_0 is given by

$$\begin{aligned}\psi_0^{-1}(w) &= \frac{1}{(\psi_0)_\#^{-1}(w)} = (J\circ(\psi_0)_\#^{-1})(w) \\ &= \exp\left(-\sum_{j\in\mathbb{Z}_+} B_j w^{j+1}\frac{d}{dw}\right)\frac{1}{w}\end{aligned}$$

where we have used Theorem 5.5.2 in the second line and the fact that

$$(\psi_0)_\#^{-1}(w) = \exp\left(-\sum_{j\in\mathbb{Z}_+} B_j w^{j+1}\frac{d}{dw}\right) w$$

by Theorem 5.5.3. Now we may compute

$$\begin{aligned}(\psi_0^{-1}\circ J)(w) &= (\psi_0^{-1})_\#(w) \\ &= \exp\left(\sum_{j\in\mathbb{Z}_+} B_j w^{-j+1}\frac{d}{dw}\right) w.\end{aligned}$$

Using Theorem 5.5.2, we find

$$\begin{aligned}(\psi_0^{-1}\circ J\circ\phi_i)(w) &= \exp\left(\sum_{j\in\mathbb{Z}_+} A_j w^{j+1}\frac{d}{dw}\right) a_0^{w\frac{d}{dw}} \exp\left(\sum_{j\in\mathbb{Z}_+} B_j w^{-j+1}\frac{d}{dw}\right) w \\ &= \exp\left(\sum_{j\in\mathbb{Z}_+} A_j w^{j+1}\frac{d}{dw}\right)\exp\left(\sum_{j\in\mathbb{Z}_+} C_j w^{-j+1}\frac{d}{dw}\right) a_0^{w\frac{d}{dw}} w \end{aligned} \tag{5.5.2}$$

where $C_j = a_0^{-j}B_j$ by Proposition 5.5.6.

To deal with the left hand side of (5.4.1), we need a small digression on more results in formal calculus.

5.6 Factoring formal exponentials and a solution to the sewing equation

To complete our formal study of the sewing equation, we need more general results regarding factorization of formal exponentials.

Our discussion is based upon the observation that the derivations of $\mathbb{C}[[x, x^{-1}]]$ of the form

$$L(n) = -x^{n+1}\frac{d}{dx}, \quad n \in \mathbb{Z}$$

span a Lie algebra, called the *Witt algebra*. One can check that algebra structure amounts to the Lie bracket relations

$$[L(m), L(n)] = (m-n)L(m+n)$$

for $m, n \in \mathbb{Z}$. Elements of this Lie algebra appear in the exponentials of the left hand side of the formal sewing equation, as discussed in Section 5.5.

The Witt algebra is a special case of the *Virasoro algebra*, namely the Virasoro algebra of central charge zero. More generally, the Virasoro algebra with central charge $c \in \mathbb{C}$ is the complex Lie algebra spanned by independent symbols of the form L_n for $n \in \mathbb{Z}$ and $c \in \mathbb{C}$ subject to the Lie bracket relations

$$\begin{aligned}[L_m, L_n] &= (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c, \\ [L_m, c] &= 0.\end{aligned}$$

Motivated by this observation and others, Barron, Huang, and Lepowsky [BHL] gave a result concerning formal exponentials, which has the following implication.

Theorem 5.6.1. *Let $\mathfrak{v}_c$ denote the Virasoro algebra of central charge c. Let*

$$g^+ = \sum_{j>0} A_j L_j \quad \text{and} \quad g^- = \sum_{j>0} C_j L_{-j}$$

be formal series in $\mathfrak{v}_c[[A_1, C_1, A_2, C_2, \ldots]]$ where the order of each A_j and each C_j is taken to be one. Then there are unique formal series

$$\Phi^-, \Phi^0, \Phi^+ \in \mathfrak{v}_c[[A_1, C_1, A_2, C_2, \ldots]]$$

of the form

$$\Phi^- = \sum_{j<0} \Phi_j L_j, \quad \Phi^+ = \sum_{j>0} \Phi_j L_j, \quad \text{and} \quad \Phi^0 = \Phi_0 L_0 + \Gamma c,$$

where $\Phi_j, \Gamma \in \mathbb{C}[[A_1, C_1, A_2, C_2, \ldots]]$ such that

$$e^{\Phi^-} e^{\Phi^+} e^{\Phi^0} = e^{g^+} e^{g^-}.$$

More precisely, the series Γ *is of the form*

$$\Gamma = \sum_{m>0} A_m C_m \frac{m^3 - m}{12} + \Gamma_0$$

where $\Gamma_0 \in \mathbb{C}[[A_1, C_1, A_2, C_2, \ldots]]$ *contains only terms of total order three or more with order at least one in the* A_m*'s and at least one in the* C_m*'s.*

By the theorem, there are unique formal series

$$\Psi^-, \Psi^0, \Psi^+ \in \mathbb{C}[[x, x^{-1}]]$$

of the form

$$\Psi^- = \sum_{j<0} \Psi_j x^{j+1} \frac{d}{dx}, \quad \Psi^+ = \sum_{j>0} \Psi_j x^{j+1} \frac{d}{dx}, \quad \text{and} \quad \Psi^0 = \Psi_0 x \frac{d}{dx},$$

where $\Psi_j \in \mathbb{C}$ such that

$$e^{\Psi^-} e^{\Psi^+} e^{\Psi^0} a_0^{x\frac{d}{dx}} x = \exp\left(\sum_{j\in\mathbb{Z}_+} A_j x^{j+1} \frac{d}{dx}\right) \exp\left(\sum_{j\in\mathbb{Z}_+} C_j x^{-j+1} \frac{d}{dx}\right) a_0^{x\frac{d}{dx}} x,$$

where the right hand side is taken from (5.5.2). Then using Theorem 5.5.2 and 5.5.3, it is clear that the candidates for uniformizing functions $F^{(1)}$ and $F^{(2)}$ are the formal power series in $\mathbb{C}[[x, x^{-1}]]$ given by

$$F^{(1)}(x) = e^{\Psi^-} x$$
$$F^{(2)}(x) = a_0^{-x\frac{d}{dx}} e^{-\Psi^0} e^{-\Psi^+} x.$$

In [H97], it is shown that the formal series $F^{(1)}$ and $F^{(2)}$ converge to analytic functions in the necessary domains and that they satisfy the normalizing conditions for the well-defined uniformizing function F.

Remark 5.6.2. Notice that in the above identity, the term Ψ^0 did not involve the series Γ of Theorem 5.6.1. This is because the Witt algebra spanned by the derivations $x^{n+1}\frac{d}{dx}$ is a Virasoro algebra with central charge $c = 0$. However, the notion of a geometric vertex operator algebra will involve, more generally, a representation of the Virasoro algebra, possibly with nontrivial central charge $c \neq 0$. In such a case, the series Γ will play a fundamental role. Because of this, we note that the series Γ is a series in the variables A_j and C_j with coefficients in $\mathbb{C}$, and following the notation of [H97], we write $\Gamma = \Gamma(A_j, C_j)$ and we let

$$\Gamma_t(A_j, C_j) = \sum_{m>0} t^m A_m C_m \frac{m^3 - m}{12} + \Gamma_0 \in \mathbb{C}[[t, t^{-1}, A_1, C_1, A_2, C_2, \ldots]]$$

where Γ_0 is as in Theorem 5.6.1.

The following lemma concerning the series $\Gamma = \Gamma(A_j, C_j)$ is discussed at more length in [H97]. Because the proof of this result is technical and beyond the scope of these notes, we omit a detailed discussion of the proof.

Lemma 5.6.3. *Let m be a positive integer and n a nonnegative integer. Let*

$$\mathfrak{C}_1 = \mathfrak{C}_1(\widehat{\mathbb{C}}, m, U, p, \phi)$$
$$\mathfrak{C}_2 = \mathfrak{C}_2(\widehat{\mathbb{C}}, n, V, q, \psi)$$

be two canonical spheres with tubes. Suppose that we may sew the 0-th puncture of $\mathfrak{C}_2$ to the i-th puncture of $\mathfrak{C}_1$. Let A_j and C_j be defined as in (5.5.2) and let $\Gamma_t(A_j, C_j)$ be defined as in Remark 5.6.2. Then for any $c \in \mathbb{C}$, the series $e^{-\Gamma_t(A_j,C_j)c}$ is absolutely convergent when $t = 1$.

We write the limit of the series in the lemma as

$$e^{-\Gamma(\mathfrak{C}_1,\mathfrak{C}_2)c}.$$

In the case where

$$\mathfrak{S}_1 = \mathfrak{S}_1(\widehat{\mathbb{C}}, m, U, p, \phi)$$
$$\mathfrak{S}_2 = \mathfrak{S}_2(\widehat{\mathbb{C}}, n, V, q, \psi)$$

are two spheres with tubes with $m > 0$ and $n \geq 0$ and if the i-th tube of $\mathfrak{S}_1$ can be sewn with the 0-th tube of $\mathfrak{S}_2$, then we find respective canonical representatives $\mathfrak{C}_1$ and $\mathfrak{C}_2$ and we write

$$e^{-\Gamma([\mathfrak{S}_1],[\mathfrak{S}_2])c} = e^{-\Gamma(\mathfrak{C}_1,\mathfrak{C}_2)c}.$$

5.7 Geometric vertex operator algebras

By now, we have endowed the collection of moduli spaces of spheres with tubes

$$\mathcal{K} = \bigcup_{n \in \mathbb{N}} \mathcal{K}(n)$$

with a partial operad structure (see Appendix A for information about operads and partial operads) given by sewing

$${}_i\infty_0 : \mathcal{K}(m) \times \mathcal{K}(n) \longrightarrow \mathcal{K}(m+n-1).$$

We seek to study this object algebraically, translating the notion of sewing into some sort of multiplicative operation. Following Huang [H97], we will introduce an appropriate notion of an algebra over $\mathcal{K}$ that we will call a geometric vertex operator algebra. We first need some notation.

For a $\mathbb{Z}$-graded vector space

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)}$$

over $\mathbb{C}$ with finite-dimensional homogeneous subspaces, let

$$V' = \coprod_{n\in\mathbb{Z}} V^*_{(n)}$$

denote its graded dual space, and let

$$\overline{V} = \prod_{n\in\mathbb{Z}} V_{(n)} = (V')^*$$

denote the algebraic completion of V. Let $\pi_n : V \to V_{(n)}$ denote the natural projection map. Let $\langle -, - \rangle$ denote the natural pairing of $\overline{V}$ and V'. Let

$$V[[t,t^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} v_n t^n : v_n \in V \right\}$$

denote the space of formal power series in t and t^{-1} with coefficients in V.

Let V be a $\mathbb{Z}$-graded vector space with finite-dimensional homogeneous subspaces. Let m and n be integers such that $m > 0$ and $n \geq 0$. For any positive integer i satisfying $i \leq m$, we define the *t-contraction map*

$$\begin{aligned}(\cdot\ {}_i{*}_0\ \cdot)_t : \mathrm{Hom}(V^{\otimes m}, \overline{V}) \times \mathrm{Hom}(V^{\otimes n}, \overline{V}) &\longrightarrow \mathrm{Hom}(V^{\otimes m+n-1}, \overline{V}[[t,t^{-1}]]) \\ (f,g) &\mapsto (f\ {}_i{*}_0\ g)_t\end{aligned}$$

by

$$\begin{aligned}&(f\ {}_i{*}_0\ g)_t(v_1 \otimes \cdots \otimes v_{m+n-1}) \\ &= \sum_{k\in\mathbb{Z}} f(v_1 \otimes \cdots \otimes v_{i-1} \otimes \pi_k g(v_i \otimes \cdots \otimes v_{i+n-1}) \otimes v_{i+n} \otimes \cdots \otimes v_{m+n-1}) t^k\end{aligned}$$

for all $v_1, \ldots, v_{m+n-1} \in V$.

If for arbitrary $v' \in V'$ and $v_1, \ldots, v_{m+n-1} \in V$, the series

$$\langle v', (f\ {}_i{*}_0\ g)_t(v_1 \otimes \cdots \otimes v_{m+n-1}) \rangle$$

is absolutely convergent when $t = 1$, then $(f\ {}_i{*}_0\ g)_1$ is a well-defined element of

$$\mathrm{Hom}(V^{\otimes m+n-1}, \overline{V})$$

and we define the *contraction*

$$f\ {}_i{*}_0\ g \in \mathrm{Hom}(V^{\otimes m+n-1}, \overline{V})$$

by

$$f\ {}_i{*}_0\ g = (f\ {}_i{*}_0\ g)_1.$$

We have a left action of S_n on $V^{\otimes n}$ defined by

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

for $\sigma \in S_n$ and $v_1, \ldots, v_n \in V$. This action induces a left action of S_n on $\mathrm{Hom}(V^{\otimes n}, \overline{V})$ defined by

$$\begin{aligned} S_n \times \mathrm{Hom}(V^{\otimes n}, \overline{V}) &\longrightarrow \mathrm{Hom}(V^{\otimes n}, \overline{V}) \\ (\sigma, f) &\mapsto \sigma(f) \end{aligned}$$

where

$$\sigma(f)(v_1 \otimes \cdots \otimes v_n) = f(\sigma^{-1}(v_1 \otimes \cdots \otimes v_n))$$

for $v_1, \ldots, v_n \in V$.

A *geometric vertex operator algebra* is a pair (V, ν) where V is a $\mathbb{Z}$-graded vector space

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}$$

with finite-dimensional homogeneous weight spaces and $\nu = \{\nu_n\}_{n \in \mathbb{N}}$ is a family of maps

$$\begin{aligned} \nu_n : \mathcal{K}(n) &\longrightarrow \mathrm{Hom}(V^{\otimes n}, \overline{V}) \\ [\mathfrak{S}] &\mapsto \nu_n([\mathfrak{S}]) \end{aligned}$$

satisfying the following axioms

(GVOA1) *Positive Energy Axiom:* For n sufficiently small, we have $V_{(n)} = 0$.

(GVOA2) *Grading Axiom:* If $v \in V_{(n)}$ is homogeneous of weight n, then

$$\langle v', \nu_1[(\widehat{\mathbb{C}}; \infty, 0; 1/w, aw)](v)\rangle = a^{-n}\langle v', v\rangle$$

for each $a \in \mathbb{C}^\times$ and $v' \in V'$.

(GVOA3) *Meromorphicity Axiom:* For any positive integer n, for any $v' \in V'$, $v_1, \ldots, v_n \in V$, the function

$$[\mathfrak{S}] \mapsto \langle v', \nu_n([\mathfrak{S}])(v_1 \otimes \cdots \otimes v_n) \rangle$$

on $\mathcal{K}(n)$ is meromorphic (in the sense of Section 5.3) and if z_i and z_j are the i-th and j-th punctures of the canonical representative of $\mathfrak{S}$ respectively, then for each v_i and v_j in V, there is a positive integer $N(v_i, v_j)$ such that for any $v' \in V'$ and $v_k \in V$ for $k \neq i, j$, the order of the pole $z_i = z_j$ of

$$\langle v', \nu_n([\mathfrak{S}])(v_1 \otimes \cdots \otimes v_n) \rangle$$

is less than $N(v_i, v_j)$.

(GVOA4) *Permutation Axiom:* If $\sigma \in S_n$, then for any $[\mathfrak{S}] \in \mathcal{K}(n)$, we have

$$\sigma(\nu_n([\mathfrak{S}])) = \nu_n(\sigma[(\mathfrak{S})]).$$

(GVOA5) *Sewing Axiom:* There exists a unique complex number c (called the *charge*) such that if

$$\mathfrak{S}_1 = \mathfrak{C}_1(\widehat{\mathbb{C}}, m, U, p, \phi)$$
$$\mathfrak{S}_2 = \mathfrak{C}_2(\widehat{\mathbb{C}}, n, V, q, \psi)$$

are two spheres with tubes with $m > 0$ and $n \geq 0$ and if the i-th tube of $\mathfrak{S}_1$ can be sewn with the 0-th tube of $\mathfrak{S}_2$, then

$$\nu_m([\mathfrak{S}_1])\ {}_i{*}_0\ \nu_n([\mathfrak{S}_2])$$

exists and we have

$$\nu_{m+n-1}([\mathfrak{S}_1]\ {}_i\infty_0\ [\mathfrak{S}_2]) = (\nu_m([\mathfrak{S}_1])\ {}_i{*}_0\ \nu_n([\mathfrak{S}_2]))e^{-\Gamma([\mathfrak{S}_1],[\mathfrak{S}_2])c}.$$

This completes the definition.

Let (V_1, μ) and (V_2, ν) be two geometric vertex operator algebras over $\mathbb{C}$. A *homomorphism* from V_1 into V_2 is a $\mathbb{Z}$-graded linear map $\eta : V_1 \to V_2$ such that for any $[\mathfrak{S}] \in \mathcal{K}(n)$, we have

$$\overline{\eta} \circ \mu([\mathfrak{S}]) = \nu([\mathfrak{S}]) \circ \eta^{\otimes n}$$

where

$$\overline{\eta} : \overline{V_1} \longrightarrow \overline{V_2}$$

denotes the natural extension of η.

Let **GVOA** denote the category of geometric vertex operator algebras. One can show that if $\eta : V_1 \to V_2$ is a homomorphism of geometric vertex operator algebras, then the central charges of V_1 and V_2 are equal. Thus for each complex number $c \in \mathbb{C}$, we may consider the category $\mathbf{GVOA}(c)$ of geometric vertex operator algebras with central charge c, which is a subcategory of the category **GVOA** of all geometric vertex operator algebras.

An important result of [H97] asserts that the category of vertex operator algebras is isomorphic to the category of vertex operator algebras, as developed in Chapter 2. That is, we have the following result, whose proof can be found in [H97].

Theorem 5.7.1. *For any $c \in \mathbb{C}$, there are two functors*

$$\mathcal{F}_c : \mathbf{VOA}(c) \longrightarrow \mathbf{GVOA}(c)$$

and

$$\mathcal{G}_c : \mathbf{GVOA}(c) \longrightarrow \mathbf{VOA}(c)$$

such that

$$\mathcal{F}_c \circ \mathcal{G}_c = \mathcal{I}_{\mathbf{GVOA}(c)} \quad \text{and} \quad \mathcal{G}_c \circ \mathcal{F}_c = \mathcal{I}_{\mathbf{VOA}(c)}$$

where $\mathcal{I}_{\mathcal{C}}$ denotes the identity functor on the category $\mathcal{C}$.

This theorem establishes a firm connection between the genus-zero two-dimensional conformal geometry of string interactions and the algebraic study of vertex operator algebras.

Appendix A

Operads

This appendix is included in order to make more explicit the formulation of a geometric vertex operator algebra as a suitable algebra over the partial operad of conformal equivalence classes of spheres with tubes.

An *operad* X consists of

- a family of sets $\{X(j)\}_{j\in\mathbb{N}}$,
- a family of compositions

$$\begin{aligned} \circ_i : X(k) \times X(j) &\longrightarrow X(j+j-1) \\ (a,b) &\mapsto a \circ_i b \end{aligned}$$

  for each $k \in \mathbb{Z}_+$, each $j \in \mathbb{N}$ and each i satisfying $1 \leq i \leq k$,
- an identity element $I \in X(1)$,
- for each nonnegative integer j, a left action of the symmetry group S_j on $X(j)$ (where S_0 is understood to be the trivial group)

such that the following axioms hold

(OP1) *Composition-associativity.* For each $k \in \mathbb{Z}_+$, each $j, l \in \mathbb{N}$, each positive integer i_1 satisfying $1 \leq i_1 \leq k$, each positive integer i_2 satisfying $1 \leq i_2 \leq k+j-1$, each $a \in X(k)$, each $b \in X(j)$, and each $c \in X(l)$, we have

$$(a \circ_{i_1} b) \circ_{i_2} c = \begin{cases} (a \circ_{i_2} c) \circ_{l+i_1-1} b, & i_2 < i_1 \\ a \circ_{i_1} (b \circ_{i_2-i_1+1} c), & i_1 \leq i_2 < i_1 + j \\ (a \circ_{i_2-j+1} c) \circ_{i_1} b, & i_1 + j \leq i_2. \end{cases}$$

(OP2) For each $k \in \mathbb{N}$, each i satisfying $1 \le i \le k$, and each $a \in X(k)$, we have

$$a \circ_i I = I \circ_1 a = a.$$

(OP3) For each $k \in \mathbb{Z}_+$, each $j \in \mathbb{N}$, each i satisfying $1 \le i \le k$, each $a \in X(k)$, each $b \in X(j)$, each $\sigma \in S_k$, and each $\tau \in S(j)$, we have

$$\sigma(a) \circ_i b = \sigma(\overbrace{1, \ldots, 1}^{i-1}, j, \overbrace{1, \ldots, 1}^{k-i})(a \circ_{\sigma(i)} b)$$
$$a \circ_i \tau(b) = (\overbrace{1 \oplus \cdots \oplus 1}^{i-1} \oplus \tau \oplus \overbrace{1 \oplus \cdots \oplus 1}^{k-i})(a \circ_i b)$$

If in the above definition, we assume that the composition maps $\circ_i$ are only partially defined, that is, each map $\circ_i$ takes a subset of $X(k) \times X(j)$ to $X(k+j-1)$; that all other data remain the same; that each of the expressions in (OP1) to (OP3) hold whenever both sides exist; and that the expressions in (OP2) always exist, then we call X a *partial operad*.

As an example, let V be a vector space and let W be a subspace of V. We use the notation V^j to denote the j-fold tensor product of V with itself for $j > 0$ and V^0 as a one element set. We define the endomorphism operad $\mathcal{M}_{V,W}$ in the following manner. Let $\mathcal{M}_{V,W}(j)$ denote the set of multilinear maps from V^j to V which map W^j to W. (We regard multilinear maps from the sets V^0 and W^0 sets to V and W, respectively, as maps of sets. In particular, $\mathcal{M}_{V,W}(0) = W$.) Now for $n \in \mathbb{Z}_+$, $m \in \mathbb{N}$, and a positive integer i satisfying $1 \le i \le n$, we define a composition map

$$\circ_i : \mathcal{M}_{V,W}(n) \times \mathcal{M}_{V,W}(m) \longrightarrow \mathcal{M}_{V,W}(n+m-1)$$
$$(f, g) \mapsto f \circ_i g$$

where

$$(f \circ_i g)(v_1 \otimes \cdots \otimes v_{n+m-1}) = f(v_1 \otimes \cdots \otimes v_{i-1} \otimes g(v_i \otimes \cdots \otimes v_{i+m-1}) \otimes v_{i+m} \otimes \cdots \otimes v_{n+m-1}).$$

The identity element is the identity map $\mathrm{id}_V : V^1 \to V$. We let S_n act on $\mathcal{M}_{V,W}(n)$ by

$$(\sigma(f))(v_1 \otimes \cdots \otimes v_n) = f(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})$$

for each $\sigma \in S_n$ and $f \in \mathcal{M}_{V,W}(n)$.

Let X and X' be operads. A *morphism* of operads $\phi : X \to X'$ is a sequence of S_j-equivariant maps $\{\phi_j : X(j) \to X'(j)\}_{j\in\mathbb{N}}$ such that $\phi_1(I) = I'$ and the following diagram commutes

$$\begin{array}{ccc} X(k) \times X(j) & \xrightarrow{\circ_i} & X(k+j-1) \\ \downarrow{\scriptstyle \phi_k \times \phi_j} & & \downarrow{\scriptstyle \phi_{k+j-1}} \\ X'(k) \times X'(j) & \xrightarrow{\circ_i'} & X'(k+j-1). \end{array}$$

In the case where X and X' are partial operads, we also require that the domains of the compositions for X are mapped into the domains of the compositions for X'.

If X is an operad, an *X-algebra* is a triple (V, W, ν) where V is a vector space, W is a subspace of V, and $\nu : X \to \mathcal{M}_{V,W}$ is a morphism of operads such that the subspace of V spanned by the elements of $\nu_0(X(0))$ is $\mathcal{M}_{V,W}(0) = W$. In the case where X is a partial operad, we require that the map ν be a morphism of partial operads.

The language of operads allows one to formulate a geometric vertex operator algebra in a different manner, namely as an algebra over a certain partial operad. More precisely, if $\mathcal{K} = \cup_{n\in\mathbb{N}}\mathcal{K}(n)$ denotes the collection of moduli spaces of spheres with tubes, then a geometric vertex operator algebra is a $\mathcal{K}$-algebra of the form (V, W, ν) with some additional requirements on the spaces V and W and on the collection of maps ν.

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